Black Hole Perturbation Theory: An Introduction

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I São Paulo School on Gravitational Physics

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Overview

Prefatory Matters

Greetings

$\rightarrow$ Acknowledgements

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→ Introduction

Background

- **BSc in Physics at UFABC** (2015-2021),
  Advisor: André Gustavo Scagliusi Landulfo, PhD.
- **MSc in Physics at UFABC** (2022-),
  Advisor: Roldão da Rocha Jr, PhD.

CCNH
CENTRO DE CIÊNCIAS NATURAIS E HUMANAS
Universidade Federal do ABC

Universidade Federal do ABC

Why We're Here

→ 2015:

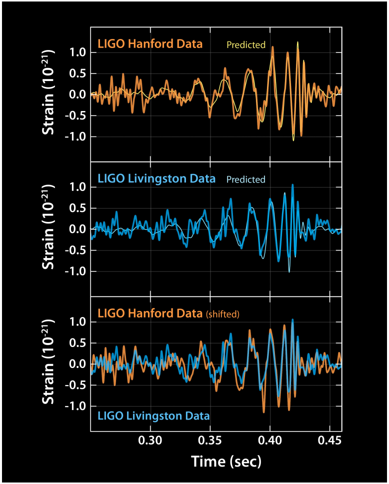

Figure: https://www.ligo.caltech.edu/image/ligo20160211a

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→ 2017 & 2018:

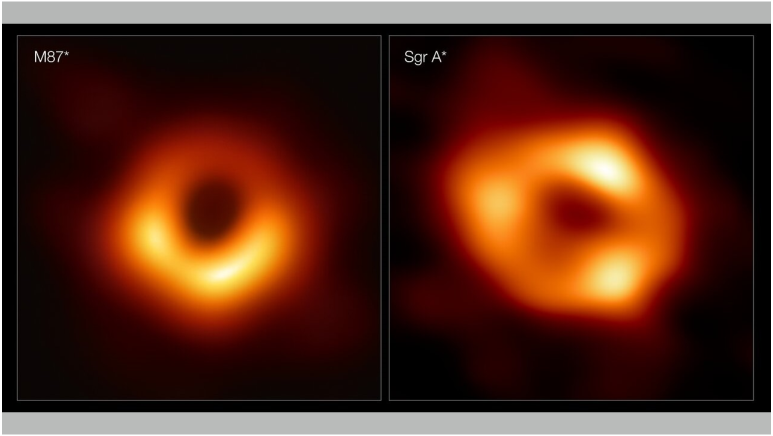

Figure: https://www.space.com/milky-way-m87-black-holes-compared-eht

The Call to Adventure

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→ Roadmap

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- Perturbation theory I

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GR Crash Course

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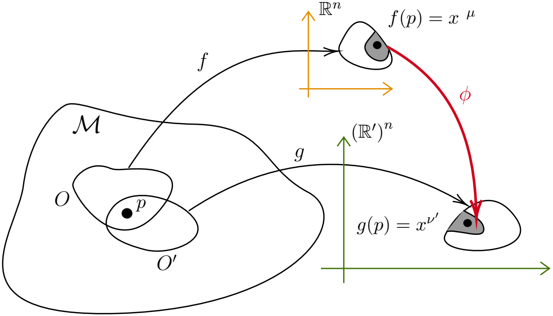
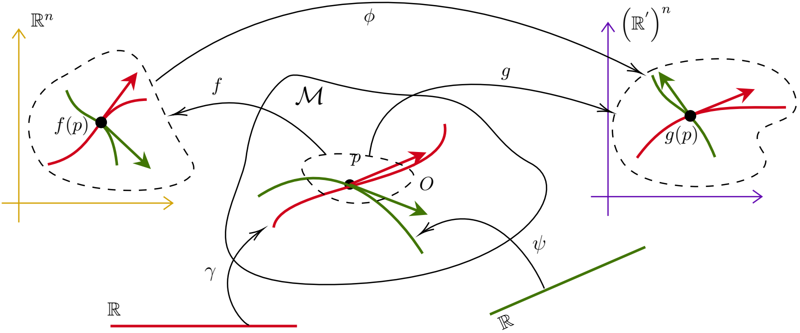


Tangent Spaces I



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$\rightarrow T_p\mathcal{M}$: Tangent space at $p \in \mathcal{M}$, i.e., the vector space which contains the tangent vectors to all curves passing through p, with basis vector ∂_μ [1].

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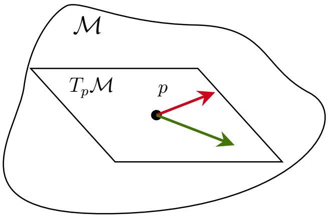

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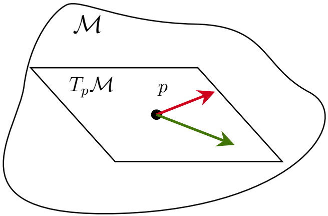

→ Introducing the tensor product operation $\otimes$, we are able to create TPSs at each $p \in \mathcal{M}$, which will contain the (k,l)-tensors of our manifold.

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$$\tilde{T}^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} = \frac{\partial\tilde{x}^{\mu_1}}{\partial x^{\sigma_1}}\frac{\partial\tilde{x}^{\mu_2}}{\partial x^{\sigma_2}}\cdots\frac{\partial\tilde{x}^{\mu_k}}{\partial x^{\sigma_k}}\frac{\partial x^{\rho_1}}{\partial\tilde{x}^{\nu_1}}\cdots\frac{\partial x^{\rho_l}}{\partial\tilde{x}^{\nu_l}}T^{\sigma_1\sigma_2\ldots\sigma_k}{}_{\rho_1\ldots\rho_l}\,. \tag{2}$$

$\rightarrow$ Such that $\tilde{\mathbf{T}} = \mathbf{T} \Rightarrow$ PGC.

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$$T_{[\mu\nu]} \equiv \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right), \tag{6}$$

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- Leibniz rule
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→ Intuitive appraoch: start from ∂_μ and fix it:

$$\nabla_\mu t^\nu = \partial_\mu t^\nu + C^\nu_{\mu\sigma} t^\sigma \tag{9}$$

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→ $C^{\nu}_{\mu\sigma}$ Does NOT transform as a tensor.

→ With $C^{\nu}_{\mu\sigma}$ we have a derivative operator that satisfies all the aforementioned requisites (see [1]).

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$\rightarrow$ One can derive (see [1]) the action of ∇ on covariant vectors given its action on scalar functions and contravariant vectors. Such that its action on (k,l)-tensors is given by:

$$\begin{aligned}\nabla_\rho V^{\mu_1\mu_2\ldots\mu_n\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_m\ldots\nu_l} &= \partial_\rho V^{\mu_1\mu_2\mu\ldots\mu_n\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_m\ldots\nu_l} + \\ &+ \sum_{n=1}^{k} \Gamma^{\mu_n}_{\rho\sigma} V^{\mu_1\mu_2\ldots\sigma\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_m\ldots\nu_l} - \\ &- \sum_{m=1}^{l} \Gamma^{\sigma}_{\rho\nu_m} V^{\mu_1\mu_2\ldots\mu_n\ldots\mu_k}{}_{\nu_1\nu_2\ldots\sigma\ldots\nu_l}\,.\end{aligned} \tag{12}$$

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→ Here we see that $\Gamma^\mu_{\nu\sigma} \Rightarrow$ Parallel transport $\Rightarrow$ Way to compare tensors at different points.

→ $\Gamma^\mu_{\nu\sigma}$ is called the Connection.

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→ $g_{\mu\nu}(x^\alpha) \Rightarrow$ dynamical notion of distances in our 4-dimensional curved spacetime:

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→ Also responsible to connect elements from $T_p\mathcal{M}$ and $(T_p\mathcal{M})^*$:

$$\begin{aligned} g_{\mu\nu} t^\mu &= t_\nu \in (T_p\mathcal{M})^* \\ g^{\mu\nu} t_\mu &= t^\nu \in T_p\mathcal{M}, \end{aligned} \tag{16}$$

→ $g^{\mu\nu}$ the inverse metric.

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Christoffel Symbols

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{18}$$

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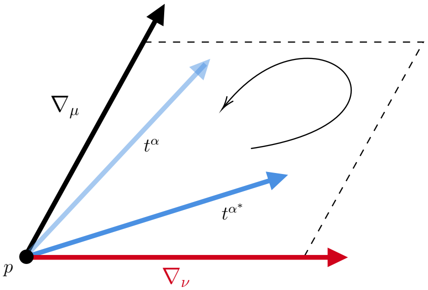
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{19}$$

$\rightarrow$ known as the Geodesic Equation.

Curvature I



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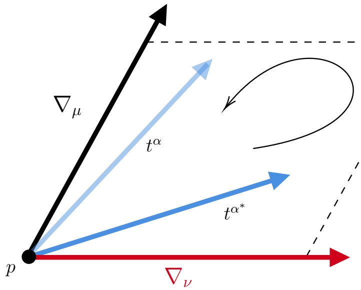

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]t^\alpha &= \nabla_\mu \nabla_\nu t^\alpha - \nabla_\nu \nabla_\mu t^\alpha \\
&= \left(\partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right) t^\sigma - 2\Gamma^\rho_{[\mu\nu]}\nabla_\rho t^\alpha \\
&= R^\alpha{}_{\sigma\mu\nu} t^\sigma - 2S^\rho{}_{\mu\nu}\nabla_\rho t^\alpha. \qquad (20)
\end{aligned}$$

Curvature II

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{R^\alpha}_{\sigma\mu\nu} \equiv \left(\partial_\mu \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\mu\sigma} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right) \tag{21}
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$\rightarrow$ We set $S^{\rho}{}_{\mu\nu} \equiv \frac{1}{2}(\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}) = \Gamma^{\rho}_{[\mu\nu]} = 0$, i.e., $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{(\mu\nu)}$.

$\rightarrow$ Hence:

$$[\nabla_{\mu}, \nabla_{\nu}]t^{\alpha} = R^{\alpha}{}_{\sigma\mu\nu}t^{\sigma}. \tag{22}$$

Curvature III

→ Properties of the Riemann tensor:

- $R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]}$.
- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$.
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→ Contracting the 1st and 3rd indices:

$$\delta^{\rho}{}_{\alpha} R^{\alpha}{}_{\nu\rho\sigma} = R^{\alpha}{}_{\nu\alpha\sigma} = R_{\nu\sigma}. \tag{23}$$

→ Such that:

Ricci Tensor

$$R_{\mu\nu} = \left(\partial_{\alpha} \Gamma^{\alpha}_{\mu\nu} - \partial_{\nu} \Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\lambda} \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\alpha}_{\nu\lambda} \Gamma^{\lambda}_{\alpha\mu} \right) \tag{24}$$

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→ Subsequently:

$$R^{\mu}{}_{\mu} = R \tag{25}$$

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→ Varying S wrt $g_{\mu\nu}$ and setting $\delta S = 0$ we get:

Einstein equations

→ The Einstein equation arises naturally from the variation of the Einstein-Hilbert action:

$$S_H = \int d^4x\sqrt{-g}R. \tag{26}$$

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Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{28}$$

where $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}}$, and $G_{\mu\nu}$ the Einstein Tensor.

Solutions

$\rightarrow$ To solve the Einstein equations is to solve for $g_{\mu\nu}$. Some notable metrics:

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Schwazrschild spacetime

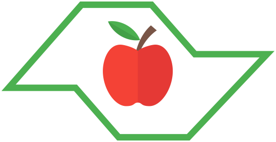
$$ds^2{}_{Sch} = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{30}$$

→ where we used $c = G = 1$.

Next Time...

Tomorrow: GR in the Weak Field limit!!!

Thank you!



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