Black Hole Perturbation Theory: An Introduction

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Overview

1. GR in the Weak Field limit

2. Gauge Symmetries in GR

3. Perturbations in Minkowski

GR in the Weak Field limit

The Weak Field Limit

→ Basic assumption:

Weak Field condition

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \tag{1}$$

→ Two possible interpretations [3, 5, 1]:

- Flat spacetime + small perturbation
- $(0, 2)$-tensor field + Minkowski background.

Perturbed Christoffel Symbols

→ Plugging the new perturbed metric into the expression for the Christoffel symbols:

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\ &= \frac{1}{2}(\eta - h)^{\rho\sigma}\left[\partial_\mu(\eta + h)_{\nu\sigma} + \partial_\nu(\eta + h)_{\mu\sigma} - \partial_\sigma(\eta + h)_{\mu\nu}\right] \\ &= \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) + O(h^2), \end{aligned} \tag{2}$$

→ where we used $\partial_\mu \eta_{\nu\gamma} = 0$, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, s.t. $g^{\mu\nu} g_{\mu\gamma} = \delta^{\nu}{}_{\gamma}$ and kept to 1^{st} order in the perturbation $h_{\mu\nu}$.

Perturbed Riemann Tensor

→ From the previous lecture:

Riemann Tensor

$$R^{\alpha}{}_{\sigma\mu\nu} \equiv \left(\partial_\mu \Gamma^{\alpha}_{\nu\sigma} - \partial_\nu \Gamma^{\alpha}_{\mu\sigma} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\right) \tag{3}$$

→ Red terms = 2^{nd} order in $h_{\mu\nu}$, hence negligeble. What will be left:

$$\begin{aligned} R^{\alpha}{}_{\sigma\mu\nu} &= \frac{1}{2}\eta^{\alpha\rho}\left[\partial_\mu\left(\partial_\nu h_{\sigma\rho} + \partial_\sigma h_{\nu\rho} - \partial_\rho h_{\nu\sigma}\right) - \partial_\nu\left(\partial_\mu h_{\sigma\rho} + \partial_\sigma h_{\mu\rho} - \partial_\rho h_{\mu\sigma}\right)\right] \\ &= \frac{1}{2}\eta^{\alpha\rho}\left(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right), \end{aligned} \tag{4}$$

where we used $[\partial_\mu, \partial_\nu] = 0$.

Perturbed Ricci Tensor & Scalar

→ From the Riemann tensor we obtain the Ricci by contraciton of the 1^{st} and 3^{rd} indices:

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial^\rho h_{\rho\nu} + \partial_\nu \partial^\rho h_{\rho\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h\right), \tag{5}$$

where $\Box = \partial_\mu \partial^\mu$, $h = h^\mu{}_\mu$.

→ And taking the trace of the Ricci:

$$\begin{aligned} R^\mu{}_\mu = R &= \frac{1}{2}\left(\partial^\mu \partial^\rho h_{\rho\mu} + \partial^\mu \partial^\rho h_{\rho\mu} - \Box h^\mu{}_\mu - \partial^\mu \partial_\mu h\right) \\ &= \partial^\mu \partial^\nu h_{\mu\nu} - \Box h, \end{aligned} \tag{6}$$

→ We now have all the ingredients for the perturbed Einstein tensor.

Perturbed Einstein Tensor

→ From previous lecture:

Einstein Tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{7}$$

→ Plugging into the above equation the expressions for the perturbed Ricci tensor, scalar and metric:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}(\eta_{\mu\nu} + h_{\mu\nu})R \\ &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R + O(h^2) \\ &= \frac{1}{2}[\partial_\mu \partial^\rho h_{\rho\nu} + \partial_\nu \partial^\rho h_{\rho\mu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \\ &\quad - \eta_{\mu\nu}(\partial^\sigma \partial^\rho h_{\sigma\rho} - \Box h)] + O(h^2). \end{aligned} \tag{8}$$

Perturbed Einstein Tensor II

→ We may introduce a new notation to simplify our expression:

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h}, \tag{9}$$

→ where $\overline{h}_{\mu\nu}$ is called the "trace-reversed" perturbation, which has $\overline{h} = -h$.

→ Subs. into the perturbed Einstein tensor and neglecting terms of $\mathcal{O}(h^2)$:

$$G^{lin}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho\overline{h}^{\rho}{}_{\nu} + \partial_\nu\partial_\rho\overline{h}^{\rho}{}_{\mu} - \Box\overline{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\sigma\partial_\rho\overline{h}^{\sigma\rho}\right), \tag{10}$$

where $G^{lin}_{\mu\nu}$ is the *linearized* Einstein tensor.

Gauge Symmetries in GR

Degrees of Freedom

$\rightarrow$ Solving Einstein equations $\Rightarrow$ Solving 10 ODEs ($G_{\mu\nu} = 0$) for 10 unknown functions $g_{\mu\nu}$?

$\rightarrow$ Bianchi identity: $\nabla^{\mu} G_{\mu\nu} = 0 \Rightarrow$ -4 degrees of freedom.

$\rightarrow$ Problem(?): 10 unknown functions and 6 equations.

$\rightarrow$ PGC $\Rightarrow g_{\mu\nu}(x^{\mu}) \rightarrow g'_{\mu\nu}(x^{\mu\prime}) \Rightarrow$ -4 degrees of freedom.

$\rightarrow$ 6 unknown func. & 6 equations!

Coordinate Transf. x Diffeomorphisms

$\rightarrow$ Analogy:

- $A^\mu \rightarrow A^\mu + \nabla^\mu \phi$ (EM gauge freedom)
- $x^\mu \rightarrow x^{\mu\prime}$ (GR gauge freedom)

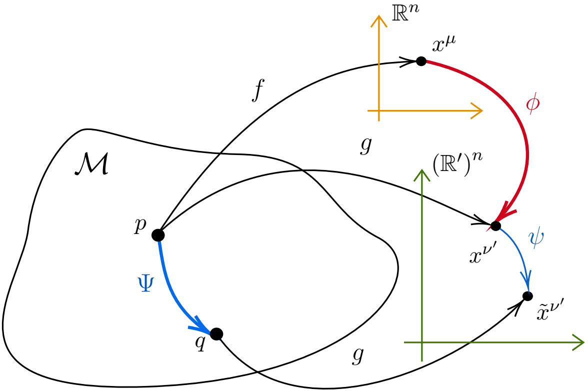
$\rightarrow$ GR's gauge freedom is sometimes referred to as GR's invariance under diffeomorphisms [3].

$\rightarrow$ Coordinate transformations x Diffeomorphisms

$$x^\mu \rightarrow x^{\nu'} \tag{11}$$

$$x^\mu \rightarrow (x')^\nu . \tag{12}$$

Coordinate Transf. x Diffeomorphisms



$$g(p) = \psi^{-1}(g(\Psi(p)))$$

$$g = \psi^{-1} \circ g \circ \Psi$$

Coordinate Transf. x Diffeomorphisms

→ This is known as a "passive" viewpoint on diffeomorphisms.[6]

→ The metric perturbation under infinitesimal diffeomorphisms transforms as:

$$\begin{aligned} h_{\mu\nu}(x^\alpha) \to h'_{\mu\nu}(x^\alpha + \xi^\alpha) &= h_{\mu\nu}(x^\alpha) + \mathcal{L}_\xi \eta_{\mu\nu} \\ &= h_{\mu\nu}(x^\alpha) + 2\nabla_{(\mu}\xi_{\nu)} \\ &= h_{\mu\nu}(x^\alpha) + 2\partial_{(\mu}\xi_{\nu)}, \end{aligned} \tag{13}$$

→ Another look at the analogy:

- $A^\mu \to A^\mu + \nabla^\mu \phi$ (EM gauge freedom)
- $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ (GR gauge freedom)

Perturbations in Minkowski

The Lorentz Gauge I

→ The vector ξ^μ allows us to get rid of unphysical degrees of freedom.

→ The trace-reversed $\overline{h}_{\mu\nu}$ transforms as [4]:

$$\overline{h}_{\mu\nu}(x^\alpha) \to \overline{h}'_{\mu\nu}(x^\alpha + \xi^\alpha) = \overline{h}_{\mu\nu}(x^\alpha) + 2\partial_{(\mu}\xi_{\nu)} - \eta_{\mu\nu}\partial^\alpha\xi_\alpha. \tag{14}$$

→ Calibrating ξ_μ we're able to set:

$$\partial^\mu \overline{h}_{\mu\nu} = 0, \tag{15}$$

→ which is known as the **Lorentz gauge**.

→ In this gauge:

$$G^{lin}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial_\rho \overline{h}^\rho{}_\nu + \partial_\nu \partial_\rho \overline{h}^\rho{}_\mu - \Box \overline{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\sigma \partial_\rho \overline{h}^{\rho\sigma}\right) \tag{16}$$

Lorentz Gauge II

→ The linearized Einstein equation becomes:

$$\Box\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{17}$$

→ Solvable by method of Green's functions.

→ Looking at this equation far from sources:

$$\Box\overline{h}_{\mu\nu} = 0 \tag{18}$$

→ Which is the famous (homogeneous) wave equation.

→ **Metric perturbation ⇒ Gravitational Waves!!!**

Gravitational Waves I

→ Ansatz:

$$\overline{h}_{\mu\nu} = \overline{P}_{\mu\nu} e^{ik_\sigma x^\sigma} \tag{19}$$

- $\overline{P}_{\mu\nu}$ = polarization tensor,
- k^μ = wave vector.

→ With this Ansatz, the wave equation implies:

$$k^\mu k_\mu = 0 \tag{20}$$

→ **GWs move at the speed of light!**

→ As Wald [6] states: GWs may be seen as massless spin-2 fields propagating in flat background.

Gravitational Waves II

→ With the same Ansatz, the Lorentz gauge implies:

$$k^\mu \overline{P}_{\mu\nu} = 0, \tag{21}$$

→ i.e., only transverse polarizations allowed (just as in EM).

→ The wave equation is linear, hence the complete solution would be:

$$\overline{h}_{\mu\nu} = \int \mathfrak{Re}\left(\overline{P}_{\mu\nu}(k^\mu) e^{i(-\omega t + k^i x_i)}\right) d^3k, \tag{22}$$

→ where we used $k^\mu \equiv (\omega, k^i)$, and $x_\mu = (-t, x_i)$, $i = 1, 2, 3$.

TT-gauge I

→ The Lorentz gauge conditions don't completely determine ξ^μ.

→ If:

$$\overline{h}_{\mu\nu} \rightarrow \overline{h}'_{\mu\nu} \rightarrow \overline{h}''_{\mu\nu} \tag{23}$$

→ For both $\overline{h}'_{\mu\nu}$ and $\overline{h}''_{\mu\nu}$ to represent the same perturbation $\overline{h}_{\mu\nu}$ we must have:

$$\Box \xi_\mu = 0 \tag{24}$$

→ Leftover gauge freedom ⇒ unphysical degrees of freedom.

→ Once more, calibrating ξ^μ we may set:

$$\overline{h} = \overline{h}_{0i} = 0, \tag{25}$$

→ known as the **Transverse-Traceless gauge**

TT-gauge II

→ Right away: $\bar{h} = 0 \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$

→ Opening the Lorentz gauge expression:

$$\partial^\mu h_{\mu 0} = \partial^0 h_{00} + \partial^i h_{i0} = 0 \tag{26}$$

$$\partial^\mu h_{\mu j} = \partial^0 h_{0j} + \partial^i h_{ij} = 0 \tag{27}$$

→ Due to $h = h_{0i} = 0$ we have:

$$\partial^0 h_{00} = 0 \tag{28}$$

$$\partial^i h_{ij} = 0 \tag{29}$$

→ Eq.28 $\Rightarrow h_{00} =$ static part (the time-**dependent** part is what matters)

→ In the Newtonian limit $h_{00} \to -2M/r$ (source), i.e. the Newtonian potential (see [2])

TT-gauge III

→ Far away from sources ⇒ $r \to \infty$, so we set $h_{00} = 0$.

→ Hence the TT-gauge produces:

$$h = h_{\mu 0} = \partial^j h_{ij} = 0 \tag{30}$$

→ Symmetries:

(i) ~~Invariance under coordinate transformations~~ (x^μ)

(ii) ~~Lorentz gauge~~ (ξ^μ)

(iii) ~~TT-gauge~~ (ξ^μ)

→ Now that we used up all our gauge freedom, what's left?

TT-gauge IV

→ Back to the solution:

$$h_{\mu\nu} = \int P_{\mu\nu}(k^{\mu})e^{ik_{\sigma}x^{\sigma}}d^3k, \tag{31}$$

→ Note that the gauges impose restrictions on $P_{\mu\nu}$:

- $k^{\mu}P_{\mu\nu} = 0$ (Lorentz gauge)
- $P^{\mu}{}_{\mu} = 0$ (Lorentz gauge)
- $P_{\mu 0} = 0$ (TT-gauge)

→ If the wave propagates in the x-direction, then:

$$k^{\mu} = (\omega, \omega, 0, 0) \tag{32}$$

→ Obs: note $k_{\mu} = (-\omega, \omega, 0, 0) \Rightarrow k^{\mu}k_{\mu} = -\omega^2 + \omega^2 = 0$, as expected.

TT-gauge V

→ Hence, $k^\mu P_{\mu\nu} = 0$ (given $P_{\mu 0} = 0$):

$$\begin{pmatrix} \omega & \omega & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P_{11} & P_{12} & P_{13} \\ 0 & P_{12} & P_{22} & P_{23} \\ 0 & P_{13} & P_{23} & P_{33} \end{pmatrix} = 0 \tag{33}$$

→ Which gives us:

$$P_{11} = P_{12} = P_{13} = 0 \tag{34}$$

→ Thus:

$$P_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P_{22} & P_{23} \\ 0 & 0 & P_{23} & -P_{22} \end{pmatrix}_{\mu\nu} , \tag{35}$$

→ where we used $P^\mu{}_\mu = 0$.

TT-gauge VI

→ Relabeling $P_{22} = p_+$ and $P_{23} = p_\times$:

$$P_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p_+ & p_\times \\ 0 & p_\times & -p_+ \end{pmatrix}_{ij}, \tag{36}$$

→ Such that, plugging it back into the equation for h_{ij} and taking the real part of $e^{ik^\mu x_\mu}$, the solution for each frequency ω of the superposition:

$$h_{ij}(x,t) = P_{ij} cos[\omega(t-x)]. \tag{37}$$

→ For details and a formal analysis of the effect of these GWs on particles, see[4, 3].

→ Here comes the intuiton...

Effect of GWs (intuiton)

→ Minkowski spacetime before perturbation (in cartesian coord.):

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{38}$$

→ Since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, turning the perturbation on:

$$ds^2 = -dt^2 + dx^2 + (1 + \Delta_+)\, dy^2 + (1 - \Delta_+)\, dz^2 + 2\Delta_\times dydz, \tag{39}$$

→ where $\Delta_+ = p_+ cos[\omega(t - x)]$ and $\Delta_\times = p_\times cos[\omega(t - x)]$

→ GW passes by ⇒ distances between points in spacetime change.

Effects of GWs (intuition) II

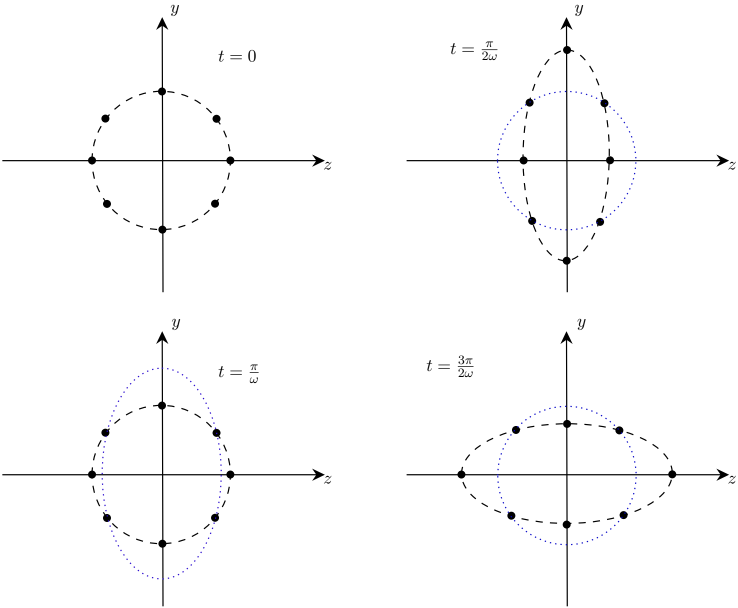
→ Set a system of particles in the yz-plane arranged in a circle centered at $x = \pi/2\omega$:

$$\Delta_+ = p_+ cos\left[\omega\left(t - \frac{\pi}{2\omega}\right)\right] \qquad \Delta_\times = p_\times cos\left[\omega\left(t - \frac{\pi}{2\omega}\right)\right]$$
$$= p_+ sin\left(\omega t\right) \qquad\qquad = p_\times sin\left(\omega t\right)$$

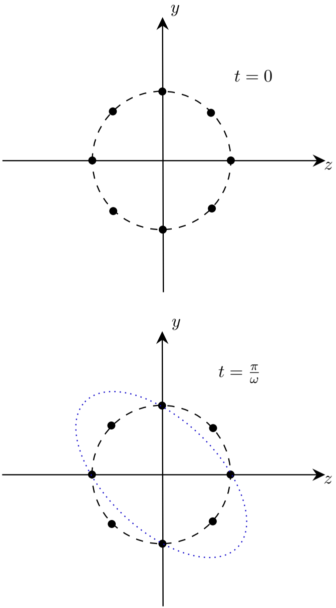
→ As time progresses and the GW passes through $x = \frac{\pi}{2\omega}$:

$$t = 0 : \quad \Delta_{+/\times} = 0$$
$$t = \frac{\pi}{2\omega} : \quad \Delta_{+/\times} = p_{+/\times}$$
$$t = \frac{\pi}{\omega} : \quad \Delta_{+/\times} = 0$$
$$t = \frac{3\pi}{2\omega} : \quad \Delta_{+/\times} = -p_{+/\times}$$

“Plus” Polarization



“Cross” Polarization



Next Time...

Tomorrow: Perturbations in Curved Spacetime!!!

Thank you!

References I

[1] Emanuele Berti. "Black Hole Perturbation Theory". In: *Summer School on Gravitational-Wave Astronomy, International Center for Theoretical Sciences, Bangalore* (2016). URL: https://www.icts.res.in/event/page/3071.

[2] Sean M Carroll. *Spacetime and geometry.* Cambridge University Press, 2019.

[3] Valeria Ferrari, Leonardo Gualtieri, and Paolo Pani. *General relativity and its applications: black holes, compact stars and gravitational waves.* CRC press, 2020.

[4] Michele Maggiore. *Gravitational waves: Volume 1: Theory and experiments.* OUP Oxford, 2007.

[5] Alexander Maloney. "General Relativity Course Notes". In: *Physics 514: General Relativity, McGill University, Montreal, Canada* (2011). URL: https://www.physics.mcgill.ca/~maloney/514/.

References II

[6] Robert M Wald. *General relativity.* University of Chicago press, 2010.