Black Hole Perturbation Theory: An Introduction

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Overview

Previously

In yesterday's lecture...

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\downarrow$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} + O(h^2)\right)$$

$$\downarrow$$

$$G^{lin}_{\mu\nu} = \frac{1}{2}(\partial_\mu\partial_\rho\overline{h}^{\rho}{}_{\nu} + \partial_\nu\partial_\rho\overline{h}^{\rho}{}_{\mu} - \Box\overline{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\sigma\partial_\rho\overline{h}^{\sigma\rho})$$

$$\downarrow$$

$$\Box\overline{h}_{\mu\nu} = 0$$

$$\downarrow$$

$$\left(-\partial_t^2 + \vec{\nabla}^2\right)\overline{h}_{\mu\nu} = 0$$

Perturbation Theory in Curved Spacetimes

General backgrounds I

→ GWs: small perturbations of a background metric $\mathring{g}_{\mu\nu}$ generated by a source $\delta T_{\mu\nu}$.

$$\begin{aligned} \mathring{T}_{\mu\nu} &\to \mathring{g}_{\mu\nu} \\ \delta T_{\mu\nu} &\to \delta g_{\mu\nu} \end{aligned} \tag{1}$$

→ Full metric:

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad |\delta g_{\mu\nu}| \ll \mathring{g}_{\mu\nu} \tag{2}$$

→ Conditions mantained by the group of infinitesimal diffeomorphisms generated by a vector ξ^μ such that:

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)}, \tag{3}$$

provided $\nabla_{(\mu}\xi_{\nu)}$ is small [2, 1].

General backgrounds II

→ From the full metric expression we get:

$$g^{\mu\nu} = \mathring{g}^{\mu\nu} - \delta g^{\mu\nu} + \mathcal{O}(h^2) \tag{4}$$

→ s.t.:

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}{}_{\sigma} + \mathcal{O}(h^2) \tag{5}$$

→ Einstein eqs.:

$$\begin{aligned} G_{\mu\nu} &= 8\pi T_{\mu\nu} \\ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 8\pi T_{\mu\nu} \end{aligned} \tag{6}$$

→ where

$$R_{\mu\nu} = \mathring{R}_{\mu\nu} + \delta R_{\mu\nu} \tag{7}$$

$$T_{\mu\nu} = \mathring{T}_{\mu\nu} + \delta T_{\mu\nu} \tag{8}$$

$$T = \mathring{T} + \delta T \tag{9}$$

→ Also remember that:

$$R_{\mu\nu} = \left(\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\alpha\mu}\right), \tag{10}$$

→ where:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{11}$$

→ Next steps:

(i) Find out who's $\delta\Gamma^\alpha_{\mu\nu}$,

(ii) Find out who's $\delta R_{\mu\nu}$.

Who's $\delta\Gamma^{\alpha}_{\mu\nu}$?

→ First we define $\Gamma_{\sigma\mu\nu}$:

$$
\begin{aligned}
\Gamma_{\sigma\mu\nu} \equiv g_{\sigma\rho}\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g_{\sigma\rho} g^{\rho\beta} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \\
&= \frac{1}{2} \delta_{\sigma}{}^{\beta} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \\
&= \frac{1}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})
\end{aligned} \tag{12}
$$

→ s.t.:

$$
\begin{aligned}
\delta\Gamma^{\rho}_{\mu\nu} &= \delta(g^{\sigma\rho}\Gamma_{\sigma\mu\nu}) \\
&= g^{\sigma\rho}\delta\Gamma_{\sigma\mu\nu} + \delta g^{\sigma\rho}\Gamma_{\sigma\mu\nu}.
\end{aligned} \tag{13}
$$

→ One may prove that[2]:

$$
\delta g^{\sigma\gamma} = -(\delta g_{\rho\mu}) g^{\sigma\rho} g^{\mu\gamma}, \tag{14}
$$

Who's $\delta\Gamma^{\alpha}_{\mu\nu}$?

$\rightarrow$ subs. into the expression for $\delta\Gamma^{\rho}_{\mu\nu}$:

$$\begin{aligned}\delta\Gamma^{\rho}_{\mu\nu} &= g^{\sigma\rho}\delta\Gamma_{\sigma\mu\nu} - (\delta g_{\gamma\alpha})g^{\alpha\rho}g^{\sigma\gamma}\Gamma_{\sigma\mu\nu} \\ &= g^{\sigma\rho}\delta\Gamma_{\sigma\mu\nu} - (\delta g_{\gamma\alpha})g^{\alpha\rho}\Gamma^{\gamma}_{\mu\nu}\end{aligned} \tag{15}$$

$\rightarrow$ changing the dummy indices $\alpha \leftrightarrow \sigma$ in the second term:

$$\begin{aligned}\delta\Gamma^{\rho}_{\mu\nu} &= g^{\sigma\rho}\delta\Gamma_{\sigma\mu\nu} - (\delta g_{\gamma\sigma})g^{\sigma\rho}\Gamma^{\gamma}_{\mu\nu} \\ &= g^{\sigma\rho}\left(\delta\Gamma_{\sigma\mu\nu} - \delta g_{\gamma\sigma}\Gamma^{\gamma}_{\mu\nu}\right)\end{aligned} \tag{16}$$

Who's $\delta\Gamma^{\alpha}_{\mu\nu}$?

$\rightarrow$ Using the definition of $\Gamma_{\sigma\mu\nu}$:

$$\delta\Gamma_{\sigma\mu\nu} = \frac{1}{2}\left[\partial_\mu(\delta g_{\nu\sigma}) + \partial_\nu(\delta g_{\mu\sigma}) - \partial_\sigma(\delta g_{\mu\nu})\right]. \tag{17}$$

$\rightarrow$ Subs. into $\delta\Gamma^{\rho}_{\mu\nu}$:

$$\begin{aligned}\delta\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\sigma\rho}\left[\partial_\mu(\delta g_{\nu\sigma}) + \partial_\nu(\delta g_{\mu\sigma}) - \partial_\sigma(\delta g_{\mu\nu}) - 2\delta g_{\gamma\sigma}\Gamma^{\gamma}_{\mu\nu}\right]\\ &= \frac{1}{2}g^{\sigma\rho}\{\left[\color{red}{\partial_\mu(\delta g_{\nu\sigma}) - \delta g_{\gamma\sigma}\Gamma^{\gamma}_{\mu\nu}}\right] + \left[\color{blue}{\partial_\nu(\delta g_{\mu\sigma}) - \delta g_{\gamma\sigma}\Gamma^{\gamma}_{\nu\mu}}\right] - \partial_\sigma(\delta g_{\mu\nu})\}\end{aligned}$$

$\rightarrow$ Note that red and blue are ALMOST covariant derivates.

Who's $\delta\Gamma^{\alpha}_{\mu\nu}$?

→ S.t.:

$$\begin{aligned}\delta\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\sigma\rho}\{\left[\nabla_{\mu}(\delta g_{\nu\sigma}) + \delta g_{\nu\gamma}\Gamma^{\gamma}_{\mu\sigma}\right] + \left[\nabla_{\nu}(\delta g_{\mu\sigma}) + \delta g_{\mu\gamma}\Gamma^{\gamma}_{\nu\sigma}\right] - \partial_{\sigma}(\delta g_{\mu\nu})\} \\ &= \frac{1}{2}g^{\sigma\rho}\{\nabla_{\mu}(\delta g_{\nu\sigma}) + \nabla_{\nu}(\delta g_{\mu\sigma}) - \left[\partial_{\sigma}(\delta g_{\mu\nu}) - \delta g_{\nu\gamma}\Gamma^{\gamma}_{\mu\sigma} - \delta g_{\mu\gamma}\Gamma^{\gamma}_{\nu\sigma}\right]\} \\ &= \frac{1}{2}g^{\sigma\rho}\{\nabla_{\mu}(\delta g_{\nu\sigma}) + \nabla_{\nu}(\delta g_{\mu\sigma}) - \nabla_{\sigma}(\delta g_{\mu\nu})\}\end{aligned}$$

→ Finally:

$$\begin{aligned}\delta\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\sigma\rho}\{\nabla_{\mu}(\delta g_{\nu\sigma}) + \nabla_{\nu}(\delta g_{\mu\sigma}) - \nabla_{\sigma}(\delta g_{\mu\nu})\} \\ &= \frac{1}{2}\mathring{g}^{\sigma\rho}\{\nabla_{\mu}(\delta g_{\nu\sigma}) + \nabla_{\nu}(\delta g_{\mu\sigma}) - \nabla_{\sigma}(\delta g_{\mu\nu})\} + \mathcal{O}(h^2) \qquad (18)\end{aligned}$$

→ $\delta\Gamma^{\rho}_{\mu\nu}$ is a tensor!!!

Who's $\delta R_{\mu\nu}$?

→ applying the same resoning:

$$\begin{aligned}\delta R_{\mu\nu} &= \partial_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \partial_\nu(\delta\Gamma^\alpha_{\alpha\mu}) + \delta(\Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\nu\mu}) - \delta(\Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\alpha\mu}) \\ &= \partial_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \partial_\nu(\delta\Gamma^\alpha_{\alpha\mu}) + (\delta\Gamma^\alpha_{\alpha\lambda})(\Gamma^\lambda_{\nu\mu}) - (\delta\Gamma^\alpha_{\nu\lambda})(\Gamma^\lambda_{\alpha\mu}) \\ &\qquad + (\Gamma^\alpha_{\alpha\lambda})(\delta\Gamma^\lambda_{\nu\mu}) - (\Gamma^\alpha_{\nu\lambda})(\delta\Gamma^\lambda_{\alpha\mu}) \end{aligned} \tag{19}$$

→ rearranging and identifying terms:

$$\delta R_{\mu\nu} = \left[\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) + \Gamma^\lambda_{\nu\alpha}(\delta\Gamma^\alpha_{\mu\lambda})\right] - \left[\nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha}) + \Gamma^\lambda_{\nu\alpha}(\delta\Gamma^\alpha_{\mu\lambda})\right] \tag{20}$$

→ The 2^{nd} and 4^{th} terms cancel, and we get:

$$\delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha}) \tag{21}$$

→ This is known as the **Palatini identity**.

Who's $\delta R_{\mu\nu}$?

$\rightarrow$ Using the expressions for $\delta\Gamma^{\rho}_{\mu\nu}$:

$$\begin{aligned}
\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) &= \frac{1}{2}\nabla_\alpha\{\mathring{g}^{\sigma\alpha}\left[\nabla_\mu(\delta g_{\nu\sigma}) + \nabla_\nu(\delta g_{\mu\sigma}) - \nabla_\sigma(\delta g_{\mu\nu})\right]\} \\
&= \frac{1}{2}\mathring{g}^{\sigma\alpha}\nabla_\alpha\left[\nabla_\mu(\delta g_{\nu\sigma}) + \nabla_\nu(\delta g_{\mu\sigma}) - \nabla_\sigma(\delta g_{\mu\nu})\right] \\
&= \frac{1}{2}\nabla^\sigma\left[\nabla_\mu(\delta g_{\nu\sigma}) + \nabla_\nu(\delta g_{\mu\sigma}) - \nabla_\sigma(\delta g_{\mu\nu})\right]
\end{aligned}$$

$$\begin{aligned}
\nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha}) &= \frac{1}{2}\nabla_\nu\{\mathring{g}^{\sigma\alpha}\left[\nabla_\mu(\delta g_{\alpha\sigma}) + \nabla_\alpha(\delta g_{\mu\sigma}) - \nabla_\sigma(\delta g_{\mu\alpha})\right]\} \\
&= \frac{1}{2}\nabla_\nu\left[\nabla_\mu(\delta g^\sigma{}_\sigma) + \nabla^\sigma(\delta g_{\mu\sigma}) - \nabla^\sigma(\delta g_{\mu\sigma})\right] \\
&= \frac{1}{2}\nabla_\nu\nabla_\mu(\delta g^\sigma{}_\sigma)
\end{aligned}$$

Who's $\delta R_{\mu\nu}$?

$\rightarrow$ Hence:

$$\delta R_{\mu\nu} = \frac{1}{2}[\nabla^{\sigma}\nabla_{\mu}(\delta g_{\nu\sigma}) + \nabla^{\sigma}\nabla_{\nu}(\delta g_{\mu\sigma}) - \nabla^{\sigma}\nabla_{\sigma}(\delta g_{\mu\nu}) - \nabla_{\nu}\nabla_{\mu}(\delta g^{\sigma}{}_{\sigma})]$$

$\rightarrow$ We may identify $\delta g_{\mu\nu} = h_{\mu\nu}$ so it becomes more friendly:

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_{\sigma}\nabla_{\mu}h_{\nu}{}^{\sigma} + \nabla_{\sigma}\nabla_{\nu}h_{\mu}{}^{\sigma} - \nabla^{\sigma}\nabla_{\sigma}h_{\mu\nu} - \nabla_{\nu}\nabla_{\mu}h\right)$$

$\rightarrow$ Taking the trace:

$$\begin{aligned}\delta R^{\mu}{}_{\mu} &= \frac{1}{2}\left(\nabla^{\sigma}\nabla^{\mu}h_{\mu\sigma} + \nabla^{\sigma}\nabla^{\mu}h_{\mu\sigma} - \nabla^{\sigma}\nabla_{\sigma}h^{\mu}{}_{\mu} - \nabla^{\mu}\nabla_{\mu}h\right)\\ &= \nabla^{\alpha}\nabla^{\sigma}h_{\alpha\sigma} - \nabla^{\alpha}\nabla_{\alpha}h \qquad (22)\end{aligned}$$

Who's $\delta G_{\mu\nu}$

$\rightarrow$ Using the Einstein tensor definition (to $\mathcal{O}(h)$):

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\
&= \mathring{R}_{\mu\nu} + \delta R_{\mu\nu} - \frac{1}{2}\left(\mathring{g}_{\mu\nu} + h_{\mu\nu}\right)\left[\left(\mathring{g}^{\alpha\beta} - h^{\alpha\beta}\right)\left(\mathring{R}_{\alpha\beta} + \delta R_{\alpha\beta}\right)\right] \\
&= \mathring{R}_{\mu\nu} + \delta R_{\mu\nu} - \frac{1}{2}\left(\mathring{g}_{\mu\nu} + h_{\mu\nu}\right)\left(\mathring{R} + \mathring{g}^{\alpha\beta}\delta R_{\alpha\beta} - h^{\alpha\beta}\mathring{R}_{\alpha\beta}\right) \\
&= \left(\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\mathring{R}\right) + \delta R_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\left(\mathring{g}^{\alpha\beta}\delta R_{\alpha\beta} - h^{\alpha\beta}\mathring{R}_{\alpha\beta}\right) - \frac{1}{2} h_{\mu\nu}\mathring{R} \\
&= \mathring{G}_{\mu\nu} + \delta G_{\mu\nu}, \qquad (23)
\end{aligned}
$$

$\rightarrow$ where:

$$
\delta G_{\mu\nu} \equiv \delta R_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\left(\mathring{g}^{\alpha\beta}\delta R_{\alpha\beta} - h^{\alpha\beta}\mathring{R}_{\alpha\beta}\right) - \frac{1}{2} h_{\mu\nu}\mathring{R} \qquad (24)
$$

Perturbed Einstein Equations I

→ For vacuum solutions as background metric:

$$\mathring{T}_{\mu\nu} = \mathring{T} = 0 \Rightarrow \mathring{R}_{\mu\nu} = \mathring{R} = 0, \tag{25}$$

→ Hence the Einstein equations become:

$$\begin{aligned} \delta G_{\mu\nu} &= 8\pi\delta T_{\mu\nu} \\ \delta R_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\left(\mathring{g}^{\alpha\beta}\delta R_{\alpha\beta}\right) &= 8\pi\delta T_{\mu\nu}, \end{aligned} \tag{26}$$

→ which are (finally) the perturbed Einstein equations!!

→ Ex.: Schwarzschild spacetime, where $\mathring{R}_{\mu\nu} = R^{Sch}_{\mu\nu} = 0$

Tensor Spherical Harmonics & The Harmonic Decomposition

The Harmonic Decomposition I

→ To fully take advantage of the problem's spherical symmetry, we'll introduce scalar, vector and tensor spherical harmonics to separate radial and angular dependencies.

→ First we must decompose our spacetime manifold:

$$\underset{(t,r)}{\mathcal{M} = \mathcal{M}_2} \times \underset{(\theta,\phi)}{\mathbb{S}^2} \tag{27}$$

→ s.t.:

$$x^\mu = (z^A, y^a), \qquad z^A = (t, r),\ y^a = (\theta, \phi), \tag{28}$$

→ which means:

$$T_p\mathcal{M} = T_p\mathcal{M}_2 \otimes T_p\mathbb{S}^2, \quad \forall p \in \mathcal{M} \tag{29}$$

The Harmonic Decomposition II

→ Vectors:

$$t^{\mu} = (t^{A}, t^{a}) \tag{30}$$

→ (0,2)-tensors:

$$t_{\mu\nu} = \begin{pmatrix} t_{AB} & t_{Aa} \\ t_{aA} & t_{ab} \end{pmatrix}_{\mu\nu} \tag{31}$$

→ So on, and so forth...

→ Demands:

- Scalar (spin-0) perturbations ⇒ Y^{lm}
- Tensor (spin-2) perturbations ⇒ $\left(Y^{lm}, Y_{a}^{lm}, S_{a}^{lm}, Z_{ab}^{lm}, S_{ab}^{lm}\right)$

→ Let's enter $\mathbb{S}^2$, i.e., the realm of the tensor spherical harmonics...

Scalar Spherical Harmonics

→ Given γ_{ab} the metric on the 2-sphere $\mathbb{S}^2$:

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & sin^2\theta \end{pmatrix}_{ab} \tag{32}$$

→ ∃ a covariant derivative $\tilde{\nabla}$ s.t.:

$$\tilde{\nabla}_a \gamma_{bc} = 0 \tag{33}$$

→ Y^{lm} = eigenvectors of the Laplacian op.:

$$\begin{aligned} \mathbb{L}Y^{lm} &= \gamma^{ab}\tilde{\nabla}_a\tilde{\nabla}_b Y^{lm} \\ &= -l(l+1)Y^{lm} \end{aligned} \tag{34}$$

Vector Spherical Harmonics

$\rightarrow$ Vectors:

- Polar/Even:

$$Y_a^{lm} \equiv (\tilde{\nabla}_\theta Y^{lm}, \tilde{\nabla}_\phi Y^{lm}) = (\partial_\theta Y^{lm}, \partial_\phi Y^{lm}) \tag{35}$$

- Axial/Odd

$$S_a^{lm} \equiv -\varepsilon_a{}^b \tilde{\nabla}_b Y^{lm} = \left(-\frac{1}{sin\theta} \partial_\phi Y^{lm}, sin\theta \partial_\theta Y^{lm} \right) \tag{36}$$

$\rightarrow$ where

$$\begin{aligned} \varepsilon_{ab} &= \sqrt{\gamma}\epsilon_{ab} \\ &= sin\theta\epsilon_{ab} \end{aligned} \tag{37}$$

Tensor Spherical Harmonics

→ $(0,2)$-Tensors:

- Polar/Even:

$$Z^{lm}_{ab} \equiv \tilde{\nabla}_a \tilde{\nabla}_b Y^{lm} + \frac{l(l+1)}{2} \gamma_{ab} Y^{lm} = \frac{1}{2} \begin{pmatrix} W^{lm} & X^{lm} \\ X^{lm} & -sin^2\theta W^{lm} \end{pmatrix} \quad (38)$$

- Axial/Odd

$$S^{lm}_{ab} \equiv \tilde{\nabla}_{(a} S^{lm}_{b)} = \frac{1}{2} \begin{pmatrix} -\frac{1}{sin\theta} X^{lm} & sin\theta W^{lm} \\ sin\theta W^{lm} & sin\theta W^{lm} \end{pmatrix} \quad (39)$$

→ where

$$X^{lm} \equiv 2 \left(\partial_\theta \partial_\phi Y^{lm} - cot\theta \partial_\phi Y^{lm} \right) \quad (40)$$

$$W^{lm} \equiv \partial^2_\theta Y^{lm} - cot\theta \partial_\theta Y^{lm} - \frac{1}{sin\theta} \partial^2_\phi Y^{lm} \quad (41)$$

→ Symmetric:

$$Z_{ab}^{lm} = Z_{(ab)}^{lm} \tag{42}$$

$$S_{ab}^{lm} = S_{(ab)}^{lm} \tag{43}$$

→ Traceless:

$$\gamma^{ab} Z_{ab}^{lm} = \gamma^{ab} S_{ab}^{lm} = 0 \tag{44}$$

→ The following properties will be the most handy:

Orthogonality Properties

→ Scalar product on $\mathbb{S}^2$:

$$\langle f, g \rangle = \int (f^* g) d\Omega \tag{45}$$

→ Scalar:

$$\langle Y^{lm}, Y^{l'm'} \rangle = \int (Y^{l'm'})^* (Y^{lm}) d\Omega = \delta^{ll'} \delta^{mm'} \tag{46}$$

→ Vector:

$$\gamma^{ab} \langle Y_a^{lm}, Y_b^{l'm'} \rangle = \gamma^{ab} \langle S_a^{lm}, S_b^{l'm'} \rangle = \int \gamma^{ab} (\partial_a Y^{lm})^* \partial_b Y^{lm} d\Omega \tag{47}$$

$$= l(l+1) \delta^{ll'} \delta^{mm'} \tag{48}$$

→ Tensor:

$$\gamma^{ab} \langle Z_{ab}^{lm}, Z_{ab}^{l'm'} \rangle = \gamma^{ab} \langle S_{ab}^{lm}, S_{ab}^{l'm'} \rangle = l(l+1)(l+2) \delta^{ll'} \delta^{mm'} \tag{49}$$

Tensor Spherical Harmonics as Basis

→ Note that wrt the metric γ_{ab}:

- t_{AB} = scalar
- t_{aA} = vector
- t_{ab} = $(0,2)$-tensor

→ Hence, we may construct a basis with the tensor spherical harmonics:

- $\{Y^{lm}\}$ = complete basis for scalar in $\mathbb{S}^2$
- $\{Y_a^{lm}, S_a^{lm}\}$ = complete basis for vectors in $\mathbb{S}^2$
- $\{Z_{ab}^{lm}, S_{ab}^{lm}\}$ = complete basis for 2-tensors in $\mathbb{S}^2$ (symmetric and traceless)

→ However

$$T_{ab} = T_{ab}^{traceless} + \frac{1}{4}\gamma_{ab}\left(\gamma^{cd}T_{cd}\right) \tag{50}$$

Tensor Spherical Harmonics as Basis

$\rightarrow \{Z_{ab}^{lm}, S_{ab}^{lm}\}$ take care of $T_{ab}^{traceless}$.

$\rightarrow \{Y^{lm}\}$ take care of $\gamma^{cd}T_{cd}$.

$\rightarrow$ Hence we have a complete basis for all scalars, vectors and symmetric 2-tensors in $\mathbb{S}^2$!!

$\rightarrow$ The set $\{Y^{lm}, Y_a^{lm}, S_a^{lm}, Z_{ab}^{lm}, S_{ab}^{lm}\}$ can be subdivided as seen previously:

- Even $= \{Y^{lm}, Y_a^{lm}, Z_a^{lm}\}$
- Odd $= \{S_a^{lm}, S_{ab}^{lm}\}$

Tensor Spherical Harmonics as basis

→ Subdivision based on how they transform under parity:

$$\theta \to \theta - \pi$$
$$\phi \to \phi + \pi$$

- Even:

$$Y^{lm}(\theta', \phi') = (-1)^l Y^{lm}(\theta, \phi) \tag{51}$$

- Odd:

$$S^{lm}(\theta', \phi') = (-1)^{l+1} S^{lm}(\theta, \phi) \tag{52}$$

The Harmonic Decomposition III

$\rightarrow \mathcal{M} = \mathcal{M}_2 \times \mathbb{S}^2$, st.:

$$h_{\mu\nu} = \begin{pmatrix} h_{AB} & h_{Aa} \\ h_{aA} & h_{ab} \end{pmatrix}_{\mu\nu} \tag{53}$$

$\rightarrow$ Remember that wrt the metric γ_{ab}:

- h_{AB} = scalar
- h_{aA} = vector
- h_{ab} = $(0,2)$-tensor

$\rightarrow$ wrt to (t, r):

- h_{AB} = $(0,2)$-tensor
- h_{aA} = vector
- h_{ab} = scalar

The Harmonic Decomposition IV

→ Where:

$$h_{AB} = \sum_{l,m} h_{AB}^{lm} Y^{lm} \tag{54}$$

$$h_{Aa} = \sum_{l,m} \left(h_A^{e,lm} Y_a^{lm} + h_A^{o,lm} S_a^{lm} \right) \tag{55}$$

$$h_{ab} = \sum_{l,m} \left[r^2 \left(K^{lm} \gamma_{ab} Y^{lm} + G^{lm} Z_{ab}^{lm} + 2h^{lm} S_{ab}^{lm} \right) \right] \tag{56}$$

→ $\{h_{AB}^{lm}, h_A^{e,lm}, h_A^{o,lm}, K^{lm}, G^{lm}, h^{lm}\}$ are all functions of (t, r) to be determined by the perturbed Einstein equations.

→ The subdivision:

- Odd: $\{h_A^{o,lm}, h^{lm}\}$
- Even: $\{h_A^{e,lm}, h_{AB}^{lm}, K^{lm}, G^{lm}\}$

Next Time...

Tomorrow: The Regge-Wheeler Equation!!!

Thank you!

References I

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