Black Hole Perturbation Theory: An Introduction

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Overview

1. Perturbations in Schwarzschild

Perturbations in Schwarzschild

Previously...

→ From previous lecture:

- Odd: $\{h_A^{o,lm}, h^{lm}\}$
- Even: $\{h_A^{e,lm}, h_{AB}^{lm}, K^{lm}, G^{lm}\}$

→ Which means that we can decompose the perturbation thusly:

$$h_{\mu\nu}(t,r,\theta,\phi) = h^e_{\mu\nu}(t,r,\theta,\phi) + h^o_{\mu\nu}(t,r,\theta,\phi), \tag{1}$$

→ where:

$$\begin{aligned} h^e_{\mu\nu} &= \sum_{l,m} \left(r^2 K^{lm} \gamma_{ab} Y^{lm} + h_{AB}^{lm} Y^{lm} + h_A^{e,lm} Y_a^{lm} + G^{lm} Z_{ab}^{lm} \right)_{\mu\nu} \\ h^o_{\mu\nu} &= \sum_{l,m} \left(h_A^{o,lm} S_a^{lm} + 2 h^{lm} S_{ab}^{lm} \right)_{\mu\nu} \end{aligned} \tag{2}$$

Perturbations $l \geq 2$

$\rightarrow$ Note that we're taking first and second derivatives of Y^{lm}, which are composed of Assoc. Legendre polynomials P_l^m:

$$\partial_a Y^{lm} \Rightarrow P_{l-1}^m$$
$$Z_{ab}^{lm} \Rightarrow P_{l-2}^m \tag{3}$$

$\rightarrow$ Hence:

$$\exists\ S_a^{lm}, Y_a^{lm} \Leftrightarrow l \geq 1$$
$$\exists\ S_{ab}^{lm}, Z_{ab}^{lm} \Leftrightarrow l \geq 2 \tag{4}$$

$\rightarrow$ $l = 0, 1$ have nothing to do with GWs:

$$monopole \rightarrow l = 0 \Rightarrow BH\ mass$$
$$dipole \rightarrow l = 1 \Rightarrow BH\ ang.\ momentum$$
$$quadrupole \rightarrow l = 2 \Rightarrow Grav.\ radiation$$

The Schwarzschild background

→ The geometry for the exterior of a spherically symmetric body of radius R is given by the Schwarzschild metric, in spherical coordinates:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{5}$$

→ where $f(r) = \left(1 - \frac{2M}{r}\right)$, $2M = R_s$.

→ For this to be a black hole:

$$R < R_s \tag{6}$$

→ Also where:

$$d\Omega = d\theta^2 + sin^2\theta d\phi^2 \tag{7}$$

The Regge-Wheeler Gauge

→ Once again we turn to a gauge transformation for simplification.

→ We perform an nfinitesimal diffeomorphism:

$$h'_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}, \tag{8}$$

→ s.t. we can set:

$$h_A^{e,lm} = G_{lm} = h_{lm} = 0, \tag{9}$$

→ which means we can make $h_A^{o,lm} \equiv h_A^{lm} = \big(h_0^{lm}(t,r), h_1^{lm}(t,r)\big)$

The Regge-Wheeler Gauge II

→ In this gauge:

$$h_{AB}^{lm} = \begin{pmatrix} fH_0^{lm} & H_1^{lm} \\ H_1^{lm} & f^{-1}H_2^{lm} \end{pmatrix} \tag{10}$$

$$h_{Aa}^{lm} = h_{aA}^{lm} = \begin{pmatrix} h_0^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_0^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \\ h_1^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_1^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \end{pmatrix} \tag{11}$$

$$h_{ab}^{lm} = \begin{pmatrix} r^2K^{lm}Y^{lm} & 0 \\ 0 & r^2(sin^2\theta)K^{lm}Y^{lm} \end{pmatrix} \tag{12}$$

The Regge-Wheeler Gauge III

→ Putting them all together:

$$h_{\mu\nu} = \begin{pmatrix} fH_0^{lm}Y^{lm} & H_1^{lm}Y^{lm} & h_0^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_0^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \\ H_1^{lm}Y^{lm} & f^{-1}H_2^{lm}Y^{lm} & h_1^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_1^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \\ * & * & r^2K^{lm}Y^{lm} & 0 \\ * & * & 0 & r^2(sin^2\theta)K^{lm}Y^{lm} \end{pmatrix}$$

→ Which renders the full metric perturbation tensor in the Regge-Wheeler gauge!

→ However, we've seen that we can decompose $h_{\mu\nu}$ based on parity:

The Regge-Wheeler gauge IV

→ Odd (Axial):

$$h_{lm}^{o} = \begin{pmatrix} 0 & 0 & h_0^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_0^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \\ 0 & 0 & h_1^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_1^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \tag{13}$$

→ Even (Polar)

$$h_{lm}^{e} = \begin{pmatrix} fH_0^{lm} & H_1^{lm} & 0 & 0 \\ H_1^{lm} & f^{-1}H_2^{lm} & 0 & 0 \\ 0 & 0 & r^2K^{lm} & 0 \\ 0 & 0 & 0 & r^2sin^2\theta K^{lm} \end{pmatrix} Y^{lm} \tag{14}$$

The Path to Regge-Wheeler I

$\rightarrow$ From this:

$$h^{o}_{lm} = \begin{pmatrix} 0 & 0 & h_0^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_0^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \\ 0 & 0 & h_1^{lm}\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & h_1^{lm}\left(sin\theta\partial_\theta Y^{lm}\right) \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \tag{15}$$

$\rightarrow$ We can reconstruct the line element substituting the perturbation components (for fixed $\{l, m\}$):

$$\begin{aligned} ds^2_{odd} = ds^2_{Schw} &- \left(h_0^{lm}\frac{1}{sin\theta}\partial_\phi Y^{lm}\right)2dtd\theta + \left(h_0^{lm} sin\theta\partial_\theta Y^{lm}\right)2dtd\phi - \\ &- \left(h_1^{lm}\frac{1}{sin\theta}\partial_\phi Y^{lm}\right)2drd\theta + \left(h_1^{lm} sin\theta\partial_\theta Y^{lm}\right)2drd\phi \end{aligned} \tag{16}$$

The Path to Regge-Wheeler II

→ And from this:

$$h^e_{lm} = \begin{pmatrix} fH_0^{lm} & H_1^{lm} & 0 & 0 \\ H_1^{lm} & f^{-1}H_2^{lm} & 0 & 0 \\ 0 & 0 & r^2K^{lm} & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta K^{lm} \end{pmatrix} Y^{lm} \tag{17}$$

→ We get the even line element (for fixed $\{l, m\}$):

$$\begin{aligned} ds^2_{even} = -f\left(1 + H_0^{lm}Y^{lm}\right)dt^2 + f^{-1}\left(1 + H_2^{lm}Y^{lm}\right)dr^2 + \\ + \left(1 + K^{lm}Y^{lm}\right)r^2 d\Omega^2 + 2H_1^{lm}Y^{lm}dtdr \end{aligned} \tag{18}$$

The Path to Regge-Wheeler III

→ Now the idea is to construct:

$$\delta G_{\mu\nu}^{even} = \delta R_{\mu\nu}^{even} - \frac{1}{2}\mathring{g}_{\mu\nu}\left(\mathring{g}^{\alpha\beta}\delta R_{\alpha\beta}^{even}\right) \tag{19}$$

$$\delta G_{\mu\nu}^{odd} = \delta R_{\mu\nu}^{odd} - \frac{1}{2}\mathring{g}_{\mu\nu}\left(\mathring{g}^{\alpha\beta}\delta R_{\alpha\beta}^{odd}\right) \tag{20}$$

→ Before that, since the background spacetime is static, we may separate the time dependence through a Fourier transform:

$$h_{\mu\nu}(t, r, \theta, \phi) = \int_{-\infty}^{+\infty} \tilde{h}_{\mu\nu}(\omega, r, \theta, \phi) e^{-i\omega t} d\omega \tag{21}$$

→ Such that:

The Path to Regge-Wheeler IV

$$\tilde{h}^o_{lm} = \begin{pmatrix} 0 & 0 & \tilde{h}^{lm}_0\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & \tilde{h}^{lm}_0\left(sin\theta\partial_\theta Y^{lm}\right) \\ 0 & 0 & \tilde{h}^{lm}_1\left(-\frac{1}{sin\theta}\partial_\phi Y^{lm}\right) & \tilde{h}^{lm}_1\left(sin\theta\partial_\theta Y^{lm}\right) \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \tag{22}$$

$$\tilde{h}^e_{lm} = \begin{pmatrix} f\tilde{H}^{lm}_0 & \tilde{H}^{lm}_1 & 0 & 0 \\ \tilde{H}^{lm}_1 & f^{-1}\tilde{H}^{lm}_2 & 0 & 0 \\ 0 & 0 & r^2\tilde{K}^{lm} & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta\tilde{K}^{lm} \end{pmatrix} Y^{lm} \tag{23}$$

$\rightarrow$ We'll use these functions instead from now on.

The Path to Regge-Wheeler V

→ Note that, for the Odd Einstein equations:

$$ds^2_{odd} = ds^2_{Schw} - \left(h_0^{lm}\frac{1}{sin\theta}\partial_\phi Y^{lm}\right)2dtd\theta + \left(h_0^{lm}sin\theta\partial_\theta Y^{lm}\right)2dtd\phi -$$
$$- \left(h_1^{lm}\frac{1}{sin\theta}\partial_\phi Y^{lm}\right)2drd\theta + \left(h_1^{lm}sin\theta\partial_\theta Y^{lm}\right)2drd\phi \quad (24)$$

→ Perturbation terms only off-diagonal, hence:

$$\delta G^{odd}_{\mu\nu} = \delta R^{odd}_{\mu\nu} - \frac{1}{2}\mathring{g}_{\mu\nu}\left(\mathring{g}^{\alpha\beta}\delta R^{odd}_{\alpha\beta}\right)$$
$$\delta G^{odd}_{\mu\nu} = \delta R^{odd}_{\mu\nu} \quad (25)$$

→ Reference [2] uses the equations:

$$\delta R_{\mu\nu} = 0 \quad (26)$$

Source I

→ Reference [1] uses:

$$\delta G_{\mu\nu} = 0 \tag{27}$$

$$\delta G_{\mu\nu} = 8\pi \delta T_{\mu\nu} \tag{28}$$

→ Where we must prepare the source the same way we did with the metric perturbations.

$$\delta T^{odd}_{\mu\nu} = \begin{pmatrix} 0 & 0 & T^{odd}_0 S_\theta & T^{odd}_0 S_\phi \\ 0 & 0 & T^{odd}_1 S_\theta & T^{odd}_1 S_\phi \\ T^{odd}_0 S_\theta & T^{odd}_1 S_\theta & T^{odd}_s S_{\theta\theta} & T^{odd}_s S_{\theta\phi} \\ T^{odd}_0 S_\phi & T^{odd}_1 S_\phi & T^{odd}_s S_{\theta\phi} & T^{odd}_s S_{\phi\phi} \end{pmatrix} \tag{29}$$

Source II

$$\delta T^{even}_{\mu\nu} = \begin{pmatrix} T^{even}_{00} Y^{lm} & T^{even}_{01} Y^{lm} & T^{even}_{0} Y^{lm}_{2} & T^{even}_{0} Y^{lm}_{3} \\ T^{even}_{01} Y^{lm} & T^{even}_{11} Y^{lm} & T^{even}_{1} Y^{lm}_{2} & T^{even}_{1} Y^{lm}_{3} \\ T^{even}_{0} Y^{lm}_{2} & T^{even}_{1} Y^{lm}_{2} & \left(T^{even} Y^{lm} + T^{even}_{z} Z^{lm}_{\theta\theta}\right) & T^{even}_{z} Z^{lm}_{\theta\phi} \\ T^{even}_{0} Y^{lm}_{3} & T^{even}_{1} Y^{lm}_{3} & T^{even}_{z} Z^{lm}_{\theta\phi} & \left(T^{even} Y^{lm} sin^2\theta + T^{even}_{z} Z^{lm}_{\phi\phi}\right) \end{pmatrix} \tag{30}$$

→ Bear in mind that everything we do on the LHS, will also be done to the RHS of the perturbed Einstein equations, including the separation of the angular part.

→ For the Odd/Axial part, both references[2, 1] yield the same exact equations, since:

$$\delta G^{odd}_{\mu\nu} = \delta R^{odd}_{\mu\nu} \tag{31}$$

Odd/Axial Equations

→ Remember that:

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\nabla_\sigma \nabla_\mu h_\nu{}^\sigma + \nabla_\sigma \nabla_\nu h_\mu{}^\sigma - \nabla^\sigma \nabla_\sigma h_{\mu\nu} - \nabla_\nu \nabla_\mu h\right)$$

→ Decomposing

$$\delta R_{\mu\nu} = \begin{pmatrix} \delta R_{AB} & \delta R_{aA} \\ \delta R_{aA} & \delta R_{ab} \end{pmatrix} \tag{32}$$

→ Where:

Odd/Axial Equations

$$2\delta R_{AB} = \sum_{l,m} \int_{-\infty}^{+\infty} \begin{pmatrix} A_{lm}^{(0)} & A_{lm}^{(1)} \\ A_{lm}^{(1)} & A_{lm}^{(2)} \end{pmatrix} Y^{lm} e^{-i\omega t} d\omega \tag{33}$$

$$2\delta R_{Aa} = \sum_{l,m} \int_{-\infty}^{+\infty} \left(\alpha_A^{lm} Y_a^{lm} + \beta_A^{lm} S_a^{lm} \right) e^{-i\omega t} d\omega \tag{34}$$

$$2\delta R_{ab} = \sum_{l,m} \int_{-\infty}^{+\infty} \left(A_{lm}^{(3)} r^2 \gamma_{ab} Y^{lm} + s_{lm} Z_{ab}^{lm} + t_{lm} S_{ab}^{lm} \right) e^{-i\omega t} d\omega \tag{35}$$

Odd/Axial Vacuum Equations

→ where:

$$\beta_0^{lm} = f(h''_{0,lm} + i\omega h'_{1,lm}) - 2i\omega\frac{f}{r}h_1^{lm} + \left(\frac{f''}{2} + \frac{l(l+1)+f-1}{r^2}\right)h_0^{lm}$$

$$\beta_1^{lm} = f^{-1}(i\omega h'_{0,lm} - \omega^2 h_1^{lm}) - 2i\omega\frac{f^{-1}}{r}h_0^{lm} + \left(\frac{f''}{2} + \frac{(l(l+1)-f-1)}{r^2}\right)h_1^{lm}$$

$$t_{lm} = i\omega f^{-1}h_0^{lm} + fh'_{1,lm} + f'h_1^{lm}$$

→ For vacuum we set:

$$\beta_A^{lm} = 0 \tag{36}$$

$$t_{lm} = 0 \tag{37}$$

Odd/Axial Sourced Equations

→ For the sourced we need to appropriately separate the angular angular part.

→ We need to construct the orthogonality relations, for ex:

(i) Multiplying $\delta R_{A\phi}$ by $\frac{\partial_\theta Y^{*lm}}{sin\theta}$

(ii) Multiplying $\delta R_{A\theta}$ by $\frac{\partial_\phi Y^{*lm}}{sin\theta}$

(iii) Subtracting $(i) - (ii)$

(iv) Integrating in $d\Omega$:

Odd/Axial Sourced Equations

→ LHS:

$$2\int d\Omega \left(\frac{\partial_\theta Y^{*lm}}{sin\theta} \delta R_{A\phi} - \frac{\partial_\phi Y^{*lm}}{sin\theta} \delta R_{A\theta} \right) =$$
$$\sum_{l,m} \int_{-\infty}^{+\infty} \alpha_A^{lm} \int \left(\frac{Y_\theta^{lm} Y_\phi^{*lm}}{sin\theta} - \frac{Y_\phi^{*lm} Y_\theta^{lm}}{sin\theta} \right) d\Omega +$$
$$\sum_{l,m} \int_{-\infty}^{+\infty} \beta_A^{lm} \left(\gamma^{ab} \langle S_a^{lm}, S_b^{l'm'} \rangle \right) =$$
$$\sum_{l,m} \int_{-\infty}^{+\infty} \beta_A^{lm} l(l+1) \tag{38}$$

Odd/Axial Sourced Equations

→ RHS:

$$16\pi \int d\Omega \left(\frac{\partial_\theta Y^{*lm}}{sin\theta} \delta T_{A\phi} - \frac{\partial_\phi Y^{*lm}}{sin\theta} \delta T_{A\theta} \right) = -l(l+1) 16\pi \delta T_A^{odd} \quad (39)$$

→ s.t.:

$$\beta_A^{lm} = -16\pi \delta T_A^{odd} \quad (40)$$

→ i.e.:

$$f(h''_{0,lm} + i\omega h'_{1,lm}) - 2i\omega \frac{f}{r} h_1^{lm} + \left(\frac{f''}{2} + \frac{l(l+1) + f - 1}{r^2} \right) h_0^{lm} = -16\pi \delta T_0^{odd}$$

$$f^{-1}(i\omega h'_{0,lm} - \omega^2 h_1^{lm}) - 2i\omega \frac{f^{-1}}{r} h_0^{lm} + \left(\frac{f''}{2} + \frac{(l(l+1) - f - 1)}{r^2} \right) h_1^{lm} = -16\pi \delta T_1^{odd}$$

Odd/Axial Sourced Equations:

→ Analogously for t_{lm}, first we define:

$$\delta R_{-} \equiv \delta R_{\theta\theta} - \frac{\delta R_{\phi\phi}}{sin^2\theta} \tag{41}$$

→ s.t.:

$$\delta R_{-} = s_{lm} W^{lm} - \frac{t_{lm}}{sin\theta} X^{lm} \tag{42}$$

$$2\delta R_{\theta\phi} = s_{lm} X^{lm} + t_{lm} sin\theta W^{lm} \tag{43}$$

→ Thus:

$$\int d\Omega \left(\frac{W^{*lm}}{sin\theta} 2\delta R_{\theta\phi} - \frac{X^{*lm}}{sin\theta} \delta R_{-} \right) = (l-1)l(l+1)(l+2)t_{lm} \tag{44}$$

→ Doing the same analogously for the RHS of t_{lm}:

$$t_{lm} = -16\pi \delta T_s^{odd} \tag{45}$$

Odd/Axial Sourced Equations:

→ Using the definitions for β_A^{lm}, t_{lm}:

$$f(h''_{0,lm} + i\omega h'_{1,lm}) - 2i\omega\frac{f}{r}h_1^{lm} + \left(\frac{f''}{2} + \frac{l(l+1)+f-1}{r^2}\right)h_0^{lm} = -16\pi\delta T_0^{odd}$$

$$f^{-1}(i\omega h'_{0,lm} - \omega^2 h_1^{lm}) - 2i\omega\frac{f^{-1}}{r}h_0^{lm} + \left(\frac{f''}{2} + \frac{(l(l+1)-f-1)}{r^2}\right)h_1^{lm} = -16\pi\delta T_1^{odd}$$

$$i\omega f^{-1}h_0^{lm} + fh'_{1,lm} + f'h_1^{lm} = -16\pi\delta T_s^{odd}$$

→ Which are the Odd Einstein equations!!!

Next Time...

Tomorrow: The Regge-Wheeler & Zerilli Equations!!!

Thank you!

References I

[1] Emanuele Berti. “Black Hole Perturbation Theory”. In: *Summer School on Gravitational-Wave Astronomy, International Center for Theoretical Sciences, Bangalore* (2016). URL: https://www.icts.res.in/event/page/3071.

[2] Valeria Ferrari, Leonardo Gualtieri, and Paolo Pani. *General relativity and its applications: black holes, compact stars and gravitational waves.* CRC press, 2020.