Welcome to celestial holography

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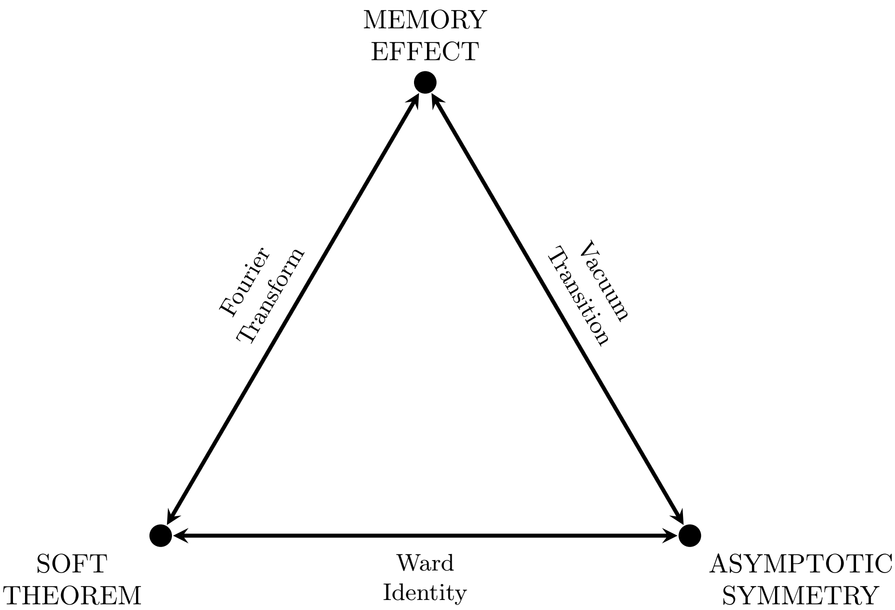
1 Lecture 1. Introduction: Infrared triangle and CFT

Celestial Holography, as follows from the name, uses the idea of holographic duality, which means establishing correspondence between symmetries in the bulk of the space-time (in our case symmetries of asymptotically flat space-time) and symmetries on its boundary (in our case 2-dimensional conformal symmetry). Therefore, to build a bridge from bulk to boundary, we need to get familiar with each of the two subjects in particular. In the first lecture we will introduce the core developments, tools and methods used to study both asymptotically flat space-times and conformal field theory.

1.1 Infrared triangle

Creation of the celestial holography program would be unimaginable without a deep study of infrared issues arising in any QFT containing massless particles that can have low energies. Before proceeding to understand the celestial basis, we should quickly talk about this IR story, and the central figure in such discussion will be the infrared triangle.



Infrared triangle is a relation that connects 3 seemingly unrelated subjects: soft theorem, asymptotic symmetries and memory effect.

Even though these three subjects seem unrelated, any two of them are deeply connected, and in reality are the same statement expressed in different "languages".

The first vertex, soft theorems, originated in QED and was generalized to gravity in 1965 by Weinberg [1]. This family of theorems describes how scattering amplitudes factorize when a massless external particle becomes soft.

The second vertex, asymptotic symmetries, is connected to symmetries that a system has on its asymptotic boundary. Back in 1962 Bondi, van der Burg, Metzner and Sachs [2, 3] expected that asymptotically flat spacetime of general relativity would have the Poincare group of special relativity. Surprisingly, they found that the symmetry group is in fact infinite dimensional.

The third vertex, memory effect, was discovered in 1974 by Zelьdovich and Polnarev [4] and developed by Christodoulou. This effect describes how a passage of gravitational wave through a LIGO-like detector produces a permanent shift in the positions of its mirrors. Though not tested yet, there are several proposals of how to test it in the next several decades.

As depicted in the picture below, we can transition between any two of these vertices. Ward identities written down for asymptotical symmetry of the S-matrix lead to the soft theorems. If we do a Fourier transform of a soft theorem, the infrared pole will become a step function in time, leading to the memory effect. And finally, transition between two vacua, related by asymptotic symmetry, is nothing else but a memory effect.

The IR physics in 4D is not the purpose of our lectures, so we will not go into too much detail, leaving just the necessary parts that will be used in the celestial story later on. For the most expanded discussion on this matter an interested reader can refer to Strominger's lectures. However, some details and explanations I will provide in this very first lecture, concentraiting on the conceptual significance of what will be done.

1.1.1 Asymptotic symmetries

The BMS group was studied back at 1960s [2,3] by H. Bondi, M. G. J. van der Burg, A. W. K. Metzner and Rainer K. Sachs to look at the behaviour of gravitational waves at null infinity, finding an infinite dimensional set of supertranslations belonging to the symmetry group. Half a century has passed and the interest to description of asymptotically flat spacetimes began to grow and it was shown in [5, 6] how supertranslation symmetry implies corresponding Ward identity and extensions of BMS group have been proposed [7,8].

To begin with, some short reminder of asymptotically flat spacetimes and their symmetries will be brought up.

Definition 1. *We will call a spacetime* ***asymptotically flat*** *if we can write its metric tensor as a sum* $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$*, where* $\eta_{\mu\nu}$ *is a Minkowski metric and* $h_{\mu\nu}$ *satisfies the following conditions:*

1. $\lim_{r\to\infty} h_{\mu\nu} = O(1/r)$;
2. $\lim_{r\to\infty} \partial_\rho h_{\mu\nu} = O(1/r^2)$;
3. $\lim_{r\to\infty} \partial_\lambda \partial_\rho h_{\mu\nu} = O(1/r^3)$,

where $r:\ r^2 = x^2 + y^2 + z^2$ *is the distance from the center of coordinates.*

Basically, these conditions mean that the curvature of spacetime vanishes at large distances from the center of coordinate system and that at large distances spacetime becomes indistinguishable from Minkowski spacetime.

One might remember how we describe the asymptotics of Minkowski spacetime when coordinates go to infinity, using Penrose diagrams. We might recall it, making a coordinate transformation:

$$\begin{aligned} t + r &= \tan\frac{1}{2}(T + R), \\ t - r &= \tan\frac{1}{2}(T - R). \end{aligned} \tag{1}$$

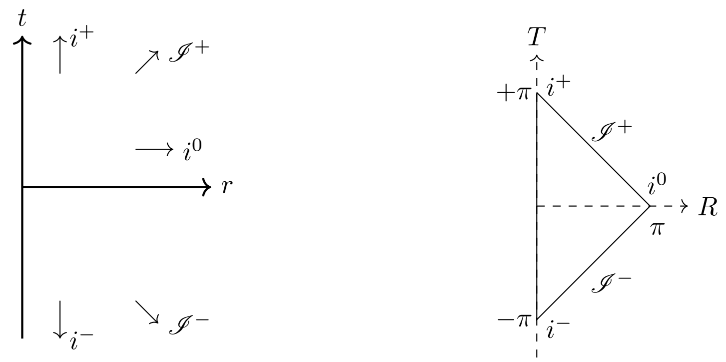


Figure 1: Minkowski space-time in spherical coordinates (t, r, θ, ϕ) and Minkowski space-time in conformally transformed spherical coordinates (T, R, θ, ϕ), where t- and r-infinities are transferred to finite distances. Here $t, r \in (-\infty, +\infty)$, and $T \in [-\pi, +\pi], R \in [0, \pi]$.

Looking at the Figure 1 we find different types on infinities on it:

1. i^+ – *timelike future infinity* with $t \to +\infty, r = const$. Massive particles who travel with speed less than the speed of light, end up at i^+.
2. i^- – *timelike past infinity* $t \to -\infty, r = const$. Massive particles who travel with speed less than the speed of light, originate from i^-.
3. i^0 – *spacelike infinity* $t = const, r \to \infty$. This is the area in which space-like slicing can be realised.
4. $\mathscr{I}^+$ – *null future infinity* $t + r \to \infty, t - r = const$. Massless particles who travel with the speed of light, end up at $\mathscr{I}^+$.
5. $\mathscr{I}^-$ – *null past infinity* $t - r \to -\infty, t + r = const$. Massless particles who travel with the speed of light, originate from $\mathscr{I}^+$.

In the following the future and past null infinities $\mathscr{I}^{\pm}$ will be the most important subjects for us of the ones listed above. Far away from the beginning of coordinates null infinities of an asymptotically flat spacetime (from where massless particles originate and where they end up) have the same structure as Minkowski spacetime. Loosely speaking, slices of these null infinities will be the boundaries where our 2D conformal field theory will dwell.

Asymptotically flat spacetime can be described in terms of a BMS formalism, starting with imposing a Bondi gauge on the form of a metric.

Definition 2. *We say that we have imposed the **Bondi gauge** if the metric meets the following conditions:*

(i) We can write metric in form

$$e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + 2g_{uA}dudx^A + g_{AB}dx^Adx^B,$$

where

$$\begin{aligned}\frac{1}{r}V(u,x^A) &= -1 + O(1/r);\\ \beta(u,x^A) &= O(1/r^2);\\ g_{uA}(u,r,x^A) &= O(1);\\ g_{AB}(u,r,x^A) &= r^2\gamma_{AB} + O(r),\end{aligned}$$

and $\gamma_{AB}:\ ds^2 = \gamma_{AB}\,dzd\bar{z} = 4\,dzd\bar{z}/(1+z\bar{z})^2$ is a unit metric on a sphere.

(ii) The conditions are satisfied:

$$g_{rr} = g_{rA} = 0 = g^{uu} = g^{uA}, \qquad \partial_r \det\big(g_{AB}/r^2\big) = 0$$

(iii) Near $\mathscr{I}^+$ we also require fall-off conditions for stress-energy tensor:

$$\begin{aligned}T^M_{uu} &= O(r^{-2}), \quad T^M_{uA} = O(r^{-2}),\\ T^M_{ur} &= O(r^{-4}), \quad T^M_{rA} = O(r^{-3}),\\ T^M_{rr} &= O(r^{-4}), \quad T^M_{AB} = O(r^{-1}).\end{aligned}$$

The flat space coordinates and Bondi coordinates are connected via

$$X^\mu = \left(u + r, r\frac{z+\bar{z}}{1+z\bar{z}}, ir\frac{z-\bar{z}}{1+z\bar{z}}, r\frac{1-z\bar{z}}{1+z\bar{z}}\right). \tag{2}$$

The stereographic coordinates $z, \bar{z}$ are related to spherical coordinates θ, φ as follows:

$$\begin{cases} z = e^{i\varphi}\cot\dfrac{\theta}{2},\\ \bar{z} = e^{-i\varphi}\cot\dfrac{\theta}{2}.\end{cases} \tag{3}$$

In the same manner, we impose the Bondi gauge on the past null infinity $\mathscr{I}^-$. Analogously to the ***retarded*** coordinate $u = t - r$ on $\mathscr{I}^+$, we introduce the ***advanced*** coordinate $v = t + r$.

A metric in the Bondi gauge can be reduced to the following form:

- Near $\mathscr{I}^+$

$$\begin{aligned}ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2&\\ -2U_zdudz - 2U_{\bar{z}}dud\bar{z} + \cdots,&\end{aligned} \tag{4}$$

  with $U_z = -\frac{1}{2}D^zC_{zz}$;

- Near $\mathscr{I}^-$

$$\begin{aligned}ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2&\\ -2V_zdvdz - 2V_{\bar{z}}dvd\bar{z} + \cdots,&\end{aligned} \tag{5}$$

  with $V_z = \frac{1}{2}D^zD_{zz}$.

Here m_B is a Bondi mass aspect (since in some spacetimes it can be connected to the mass), C_{zz} and D_{zz} are defined as outgoing and incoming shear tensors, while $N_{zz} \equiv \partial_u C_{zz}$, $M_{zz} \equiv \partial_v D_{zz}$ – outgoing and incoming Bondi news tensors. All these functions may depend on $(u, z, \bar{z})$, but not on r.

Exercise 1. *Show that in these coordinates, the Minkowski metric takes the form:*

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{6}$$

Actually, m_B, C_{zz}, N_z are not all independent. They are related by constraint equations. The uu constraint gives [9]

$$\partial_u m_B = \frac{1}{4}D_z^2 N^{zz} + \frac{1}{4}D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2}T_{uu}, \tag{7}$$

while the uz constraint reduces to

$$\begin{aligned}\partial_u N_z = \frac{1}{4}D_z\left(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\right) - T^M_{uz} + \partial_z m_B + \frac{1}{16}D_z\partial_u(C_{zz}C^{zz}) \\ - \frac{1}{4}\left(N^{zz}D_z C_{zz} + N_{zz}D_z C^{zz}\right) - \frac{1}{4}D_z\left(C^{zz}N_{zz} - N^{zz}C_{zz}\right),\end{aligned} \tag{8}$$

where we defined

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + \lim_{r\to\infty} r^2 T^M_{uu}. \tag{9}$$

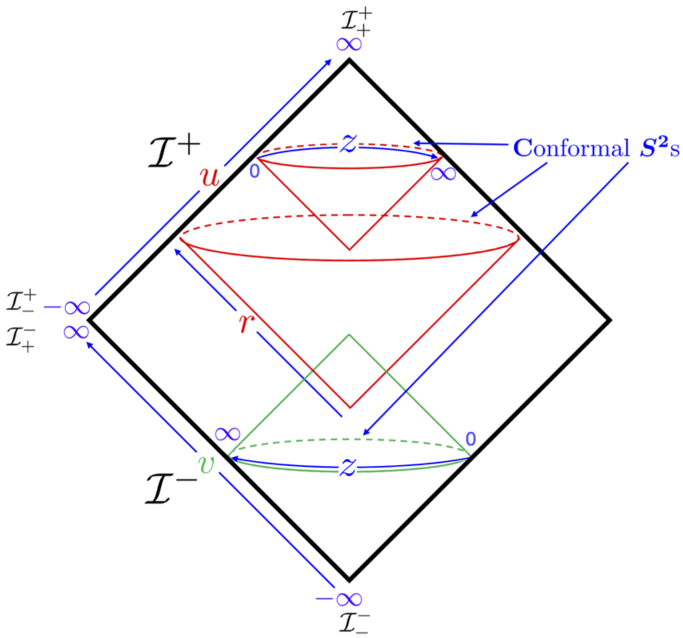


Figure 2: The structure of our asymptotically flat spacetime. $\mathscr{I}^\pm$ is denoting future and past null infinities; u and v are retarded and advanced Bondi time coordinates.

We will not perform the calculations leading to the symmetry group of (4) and its copy near $\mathscr{I}^-$ (namely, to the BMS$^\pm$ groups), as they are quite volumous. Instead, we will write down the generating vector fields, omitting the computational part.

Supertranslations near $\mathscr{I}^+$ are generated by vector fields

$$\xi^+ = f\partial_u - \frac{1}{r}(D^{\bar{z}}f\partial_z + D^z f\partial_{\bar{z}}) + D^z D_z f\partial_r, \tag{10}$$

and near $\mathscr{I}^-$ by

$$\xi^- = f^-\partial_v + \frac{1}{r}(D^{\bar{z}}f^-\partial_z + D^z f^-\partial_{\bar{z}}) - D^z D_z f^-\partial_r, \tag{11}$$

where $f(z,\bar{z}), f^-(z,\bar{z})$ – arbitary function on S^2, therefore one can clearly see that there are infinitely many supertranslations. If we take $f(z,\bar{z}) = const$, it will generate u-translations, meanwhile if we take it to be $l = 1$

harmonic on the sphere, we retrieve three spacial translations. Namely:

$$\begin{aligned} &\text{x-translations :} && f_x(z,\bar z)=\sin\theta\cos\phi=\frac{z+\bar z}{1+z\bar z}, \\ &\text{y-translations :} && f_y(z,\bar z)=\sin\theta\sin\phi=-i\frac{z-\bar z}{1+z\bar z}, \\ &\text{z-translations :} && f_z(z,\bar z)=\cos\theta=\frac{1-z\bar z}{1+z\bar z}. \end{aligned} \tag{12}$$

Exercise 2. *Show that they indeed generate 3 spatial translations.*

To calculate, for example, how metric tensor changes under these transformations, we will need to compute its Lie derivative:

$$\delta g_{\mu\nu}=\mathcal{L}_\xi g_{\mu\nu}=\xi^\rho\partial_\rho g_{\mu\nu}+(\partial_\mu\xi^\rho)g_{\rho\nu}+(\partial_\nu\xi^\rho)g_{\mu\rho} \tag{13}$$

Therefore, the original BMS group includes Lorentz transformations and supertranslations. Later it was found that rotations can also be extended to *superrotations*, what lead to the *extended BMS group*, but this is a whole different story that we will discover during the 3rd lecture.

Since we are interested in scattering processes between $\mathscr{I}^\pm$, which are not connected together by definition, we need a prescription of initial values on them of what we are integrating. In our case we integrate m_B, therefore we need to determine the initial value. For now we propose that BMS$^+$ frame should be determined by the Lorentz- and CPT-invariant matching conditions, defining $C_{zz}|_{\mathscr{I}^+_-}=2D_z^2C|_{\mathscr{I}^+_-}$

$$C(z,\bar z)|_{\mathscr{I}^+_-}=C(z,\bar z)|_{\mathscr{I}^-_+}\,,\quad m_B(z,\bar z)|_{\mathscr{I}^+_-}=m_B(z,\bar z)|_{\mathscr{I}^-_+}\,. \tag{14}$$

These conditions basically "glue together" our asymptotic data on $\mathscr{I}^-_+$ and $\mathscr{I}^+_-$. Instead of having two independent symmetry groups $BMS^\pm$, we connect them through this condition and have one.

The matching condition for N_z is discussed below. As mentioned, conditions (14) break the combined BMS$^+\times$BMS$^-$ action on $\mathscr{I}^+$ and $\mathscr{I}^-$ down to the diagonal subgroup that preserves these conditions, namely,

$$f(z,\bar z)|_{\mathscr{I}^+_-}=f(z,\bar z)|_{\mathscr{I}^-_+}\,. \tag{15}$$

These infinite number of matching conditions yield the existence of infinite number of conserved charges. The supertranslation charges at $\mathscr{I}^+$ can be written as [10, 11]

$$Q_f^+=\frac{1}{4\pi G}\int_{\mathscr{I}^+_-}d^2z\gamma_{z\bar z}fm_B, \tag{16}$$

$$Q_f^-=\frac{1}{4\pi G}\int_{\mathscr{I}^-_+}d^2z\gamma_{z\bar z}fm_B. \tag{17}$$

Integrating by parts, using the constraint equation, and assuming m_B decays to zero in the far future, we can write

$$Q_f^+=\frac{1}{16\pi G}\int dud^2zf\gamma_{z\bar z}N_{zz}N^{zz}-\frac{1}{8\pi G}\int d^2z\gamma^{z\bar z}fD_z^2D_{\bar z}^2N, \tag{18}$$

$$Q_f^-=\frac{1}{16\pi G}\int dvd^2zf^-\gamma_{z\bar z}M_{zz}M^{zz}+\frac{1}{8\pi G}\int d^2z\gamma^{z\bar z}f^-D_z^2D_{\bar z}^2. \tag{19}$$

Of course the charges $Q^\pm$ existed even before imposing matching conditions. But the fact that a conservation law $Q_f^+=Q_f^-$ takes place only if the matching conditions are set.

1.1.2 Soft theorem

The Quantum Field Theory description of scattering processes of gravitons and scalar particles was studied back in 1960s [12–15]. The original soft graviton theorem corresponding to some scattering process with a soft graviton attached to one of it's legs, published by Steven Weinberg in 1965 [1], was claiming that the resulting amplitude with soft graviton differs from the original one just by multiplying it by some factor depending on the particles' momenta with the $1/q$ dependence on the soft graviton momenta plus terms $\mathcal{O}(1)$ of it. In 2014 Cachazo and Strominger [16] extended the theorem for subleading order, calculating the second term in the soft expansion.

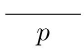
The soft theorem will be derived from Einstein-Hilbert action coupled with a massless scalar field ϕ:

$$S = -\int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \tag{20}$$

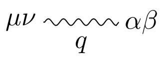
in the weak field expansion $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Imposing harmonic gauge $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$, one can expand the gravitational and scalar parts of the Lagrangian as

$$\begin{aligned}
\mathcal{L}_{\rm g} &= -\frac{2}{\kappa^2}R - \frac{1}{2}\partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h + \partial^\mu h_{\mu\nu}\partial_\rho h^{\nu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \cdots, \\
\mathcal{L}_s &= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\kappa h^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\rho}\partial_\sigma\phi\partial_\rho\phi\right] + \cdots.
\end{aligned} \tag{21}$$

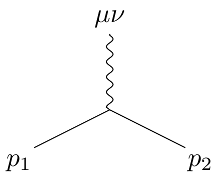
and find the Feynman rules for such theory [17]:



$$= \frac{-i}{p^2 - i\varepsilon}, \tag{22}$$

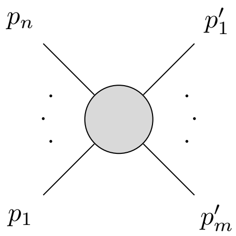


$$= \frac{-i}{2}\frac{\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}}{p^2 - i\varepsilon}, \tag{23}$$



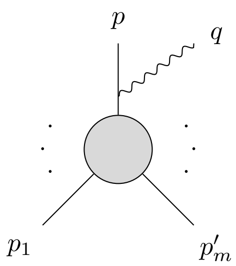
$$= \frac{i\kappa}{2}(p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \eta_{\mu\nu}p_1 \cdot p_2). \tag{24}$$

Consider a scattering process with n incoming particles of momenta $\{p_1, p_2, \ldots, p_n\}$ and m outgoing particles of momenta $\{p'_1, p'_2, \ldots, p'_m\}$:



(25)

One then add an outgoing soft (low-energy) graviton with momentum q, attached to the external line of the process Feynman diagram with momentum $p \in \{p_1, p_2, \ldots, p_n, p'_1, p'_2, \ldots, p'_m\}$.



(26)

This will give the original scattering amplitude an additional factor consisted of

- propagator with momentum $p + q$ (or $p - q$ depending on is the initial particle incoming or outgoing):

$$\frac{-i}{(2\pi)^4}\frac{1}{(p + \eta q)^2 + m^2 - i\epsilon}, \tag{27}$$

  where $\eta = \pm 1$ shows was the particle incoming or outgoing,

- vertex $p + q \to p$ (or $p \to p - q$ in case of an incoming particle):

$$\frac{i}{2}(2\pi)^4\sqrt{8\pi G}(2p^\mu+\eta q^\mu)(2p^\nu+\eta q^\nu). \tag{28}$$

For $q\to 0$ the resulting factor will be written as

$$\begin{aligned}
&\frac{-i}{(2\pi)^4}\frac{1}{(p+\eta q)^2+m^2-i\epsilon}\times\frac{i}{2}(2\pi)^4\sqrt{8\pi G}(2p^\mu+\eta q^\mu)(2p^\nu+\eta q^\nu)=\\
&=\frac{\sqrt{8\pi G}}{2}\frac{4p^\mu p^\nu+2\eta(p^\mu q^\nu+q^\mu p^\nu)+q^\mu q^\nu}{p^2+2\eta p\cdot q+q^2+m^2-i\epsilon}=\left[p^2+m^2=0\right]=\\
&=\frac{\sqrt{8\pi G}}{2}\frac{4p^\mu p^\nu}{2\eta p\cdot q-i\epsilon}+O(1)=\sqrt{8\pi G}\frac{\eta p^\mu p^\nu}{p\cdot q-i\eta\epsilon}+O(1).
\end{aligned} \tag{29}$$

This is the leading order contribution of the 1 soft graviton scattering with a $1/(p\cdot q)$ pole. Now, if attaching this soft graviton to an arbitary diagram, one has to sum this result (29) over all external lines, which will be given as:

$$\sum_{i=1}^{m+n}\sqrt{8\pi G}\frac{\eta p^\mu p^\nu}{p\cdot q-i\eta\epsilon}=\sqrt{8\pi G}\Big(\sum_{i=1}^{m}\frac{p_i'^\mu p_i'^\nu}{p_i'\cdot q-i\epsilon}-\sum_{i=1}^{n}\frac{p_i^\mu p_i^\nu}{p_i\cdot q+i\epsilon}\Big)\equiv\sqrt{8\pi G}S^{(0)\mu\nu}. \tag{30}$$

This way, the amplitude for a scattering process with one soft graviton ($q\to 0$), originally obtained in [1], can be constructed as follows, provided that the original $m+n$-point amplitude (25) is $\mathcal{A}_{m+n}$:

$$\mathcal{A}_{m+n+1\ \mathrm{s.g.}}(p_1,\dots p_m',q)=\sqrt{8\pi G}S^{(0)}_{\mu\nu}\epsilon^{\mu\nu}\mathcal{A}_{m+n}(p_1,\dots p_m')+O(1). \tag{31}$$

If one is interested in the subleading order $O(1)$ of soft graviton theorem, it was shown in [16] that

$$\mathcal{A}_{n+m+1}(p_1,\dots p_m',q)=\left(S^{(0)}+S^{(1)}\right)\mathcal{A}_{n+m}(p_1,\dots p_m',q)+\mathcal{O}(q^1), \tag{32}$$

with subleading soft factor

$$S^{(1)}\equiv -i\sum_{a=1}^{m+n}\frac{E_{\mu\nu}k_a^\mu\left(q_\rho J_a^{\rho\nu}\right)}{q\cdot k_a}. \tag{33}$$

Here arises the angular momentum factor, which was absent in the leading order.

1.1.3 Memory effect

Now we will move to describing gravitational memory effect, the final part of IR triangle. The memory effect characterizes a pair of inertial detectors stationed near $\mathscr{I}^+$ in a region with no Bondi news $N_{zz}=M_{zz}=0$ at both late and early times. At intermediate times, gravity waves pass through, causing oscillating distortions in their relative separation. This relative separation lies in a plane $(z,\bar z)$ perpendicular to the direction of gravitational wave propagation, and we will denote the corresponding positions as $(s^z,s^{\bar z})$.

As the gravitational wave propagates perpendicular to the plane of two inertial masses (detectors), it is displaced. After the gravitational wave has passed, the masses are permanently displaced, due to the memory effect.

The equation of geodesic deviation implies

$$r^2\gamma_{z\bar z}\partial_u^2 s^{\bar z}=-R_{uzuz}s^z, \tag{34}$$

where

$$R_{uzuz}=-\frac{r}{2}\partial_u^2 C_{zz}. \tag{35}$$

Now we just integrate this equation and obtain the shift:

$$\Delta s^{\bar z}=\frac{\gamma^{z\bar z}}{2r}\Delta C_{zz}s^z. \tag{36}$$

This is the gravitational memory effect. The difference ΔC_{zz} between initial and final transverse metric components need not vanish, as flatness does not require $C_{zz}=0$. Proposals to measure the gravitational memory effect with a variety of methods are given in [18, 19].

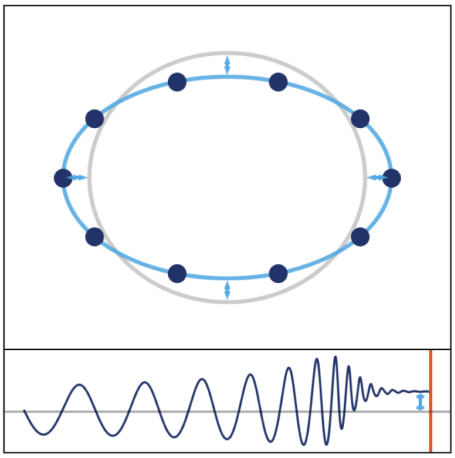


Figure 3: The top image shows how the circle of probe masses (detectors) is displaced before (grey) and after (blue) gravitational wave passes. The bottom graph depicts how the distance between two of the probe masses (detectors) starts oscillating when the gravitational wave passes through and ends up being permanently changed by some distance.

1.1.4 Soft Theorem ↔ Asymptotic Symmetry (Ward identities)

Before deriving Ward identities from soft theorem result, some time will be dedicated to understanding what is a Ward identity, which follows from supertranslation symmetry of S-matrix

$$Q_f^+ S - S Q_f^- = 0. \tag{37}$$

Firstly, one should fix some important notations. In- and out- states on the conformal sphere at past and future null infinities ($\mathcal{I}^{\pm}$) will be denoted as $|z_1^{\rm in}, \cdots\rangle$ and $\langle z_1^{\rm out}, \cdots|$, respectively .

Approaching point of $u, r \to +\infty$, the parametrization of spatial momentum is given by

$$\vec{p} = \frac{\omega}{1+z\bar{z}}(z+\bar{z}, -iz+i\bar{z}, 1-z\bar{z}), \quad \vec{p}\cdot\vec{p} = \omega^2. \tag{38}$$

Asymptotical approximation of gravitational field at $u, r \to +\infty$ can be written in terms of mode expansion as

$$h_{\mu\nu}^{\rm out}(x) = \sum_{\alpha=\pm} \int \frac{d^3 q}{(2\pi)^3 2\omega_q} \left[\varepsilon_{\mu\nu}^{\alpha *}(\vec{q}) a_\alpha^{\rm out}(\vec{q}) e^{iq\cdot x} + \varepsilon_{\mu\nu}^{\alpha}(\vec{q}) a_\alpha^{\rm out\,\dagger}(\vec{q}) e^{-iq\cdot x}\right], \tag{39}$$

where $q^0 = \omega_q = |\vec{q}|$, $\alpha = \pm$ are the two possible graviton helicities and the creation-annihilation commutation relation is taken to be

$$[a_\alpha^{\rm out}(\vec{q}), a_\beta^{\dagger {\rm out}}(\vec{q}\,')] = \delta_{\alpha\beta}(2\omega_q)(2\pi)^3\delta^3(\vec{q}-\vec{q}\,'). \tag{40}$$

Inserting annihilation operator $a_\alpha^{\rm out}(\vec{q})$ corresponds to introducing a graviton of momentum q and polarization α in terms of soft graviton theorem.

The graviton four-momentum can be parametrized as

$$q^\mu = \frac{\omega_q}{1+w\bar{w}}(1+w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1-w\bar{w}), \tag{41}$$

with polarization tensors $\varepsilon^{\pm\mu\nu} = \varepsilon^{\pm\mu}\varepsilon^{\pm\nu}$ may be written down as

$$\begin{aligned} \varepsilon^{+\mu}(\vec{q}) &= \frac{1}{\sqrt{2}}(\bar{w}, 1, -i, -\bar{w}), \\ \varepsilon^{-\mu}(\vec{q}) &= \frac{1}{\sqrt{2}}(w, 1, i, -w), \end{aligned} \tag{42}$$

obeying $\varepsilon^{\pm\mu\nu} q_\nu = \varepsilon^{\pm\mu} q_\mu = 0$ and

$$\varepsilon_z^+(\vec{q}) = \partial_z x^\mu \varepsilon_\mu^+(\vec{q}) = \frac{\sqrt{2}r\bar{z}(w-\bar{z})}{(1+z\bar{z})^2}, \quad \varepsilon_z^-(\vec{q}) = \partial_z x^\mu \varepsilon_\mu^-(\vec{q}) = \frac{\sqrt{2}r(1+w\bar{z})}{(1+z\bar{z})^2}. \tag{43}$$

From the expression for metric in Bondi coordinates near future null infinity, substituting mode expansion (39) and the definition of $h_{zz} = \partial_z x^\mu \partial_z x^\nu h_{\mu\nu}$ one can see

$$\begin{aligned} C_{zz}(u,z,\bar{z}) &= \kappa \lim_{r\to\infty} \frac{1}{r} h^{\text{out}}_{zz}(r,u,z,\bar{z}) = \\ &= \kappa \lim_{r\to\infty} \frac{1}{r} \partial_z x^\mu \partial_z x^\nu \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \left[\varepsilon^{\alpha *}_{\mu\nu}(\vec{q}) a^{\text{out}}_\alpha(\vec{q}) e^{-i\omega_q u - i\omega_q r(1-\cos\theta)} + \text{h.c.} \right], \end{aligned} \tag{44}$$

where θ is an angle between $\vec{x}$ and $\vec{q}$.

As approaching the spatial infinity $r \to \infty$, the expression under integral will oscillate with stationary points $0, \pi$, and therefore approximating over the spatial momentum the integral may be written as

$$C_{zz} = -\frac{i\kappa}{4\pi^2(1+z\bar{z})^2} \int_0^\infty d\omega_q \left[a^{\text{out}}_+(\omega_q \hat{x}) e^{-i\omega_q u} - a^{\text{out}\,\dagger}_-(\omega_q \hat{x}) e^{i\omega_q u} \right]. \tag{45}$$

Now the outgoing Bondi news tensor may be rewritten in terms of creation and annihilation operators as follows:

$$\begin{aligned} N^\omega_{zz}(z,\bar{z}) &\equiv \int_{-\infty}^\infty du e^{i\omega u} \partial_u C_{zz} = \\ &= -\frac{\kappa}{2\pi(1+z\bar{z})^2} \int_0^\infty d\omega_q \left[a^{\text{out}}_+(\omega_q \hat{x}) \delta(\omega_q - \omega) + a^{\text{out}}_-(\omega_q \hat{x})^\dagger \delta(\omega_q + \omega) \right]. \end{aligned} \tag{46}$$

If $\omega > 0$ ($\omega < 0$), only the first (second) term survives and one can define

$$\begin{aligned} N^\omega_{zz}(z,\bar{z}) &= -\frac{\kappa\omega a^{\text{out}}_+(\omega\hat{x})}{2\pi(1+z\bar{z})^2}, \\ N^{-\omega}_{zz}(z,\bar{z}) &= -\frac{\kappa\omega a^{\text{out}}_-(\omega\hat{x})^\dagger}{2\pi(1+z\bar{z})^2}; \end{aligned} \tag{47}$$

where $\omega > 0$.

The zero mode will be defined as

$$N^0_{zz} \equiv \lim_{\omega\to 0^+} \frac{1}{2}(N^\omega_{zz} + N^{-\omega}_{zz}) = -\frac{\kappa}{4\pi(1+z\bar{z})^2} \lim_{\omega\to 0^+} \left[\omega a^{\text{out}}_+(\omega\hat{x}) + \omega a^{\text{out}}_-(\omega\hat{x})^\dagger \right] \tag{48}$$

The same analysis holds for the past null infinity, where in the end one arrives to expression for incoming Bondi news tensor

$$M^0_{zz}(z,\bar{z}) \equiv -\frac{\kappa}{4\pi(1+z\bar{z})^2} \lim_{\omega\to 0^+} \left[\omega a^{\text{in}}_+(\omega\hat{x}) + \omega a^{\text{in}}_-(\omega\hat{x})^\dagger \right]. \tag{49}$$

Now one can define an operator

$$\mathcal{O}_{zz} \equiv N^0_{zz}(z,\bar{z}) + M^0_{zz}(z,\bar{z}), \tag{50}$$

which will arise a in the next section when deriving Ward identity, to merge Fock space and Bondi metric notations.

If one fix arbitary function $f(w,\bar{w}) = \frac{1}{z-w}$, the matrix element between such states can be then written as

$$\langle z^{\text{out}}_1, \cdots | : P_z S : | z^{\text{in}}_1, \cdots \rangle = \langle z^{\text{out}}_1, \cdots | S | z^{\text{in}}_1, \cdots \rangle \left[\sum_{k=1}^m \frac{E^{\text{out}}_k}{z - z^{\text{out}}_k} - \sum_{k=1}^n \frac{E^{\text{in}}_k}{z - z^{\text{in}}_k} \right], \tag{51}$$

where current P_z is given by

$$P_z \equiv \frac{1}{2G} \left(\int_{-\infty}^\infty dv \partial_v V_z - \int_{-\infty}^\infty du \partial_u U_z \right) = \frac{1}{4G} \gamma^{z\bar{z}} \partial_z \mathcal{O}_{zz} \tag{52}$$

with $\mathcal{O}_{zz}$ defined in (50) and for incoming and outgoing energies $E^{\text{in}}_k, E^{\text{out}}_k$ holds conservation law

$$\sum_{k=1}^m E^{\text{out}}_k = \sum_{k=1}^n E^{\text{in}}_k \tag{53}$$

Ward identity, corresponding to supertranslations, is given by expression (51).

Now, after understanding how supertranslation symmetry of asumptotically flat spacetimes is leading to Ward identity, one may be interested in finding connection between it and soft theorem. To do so, some preparation may be needed.

First of all it is useful to write down momenta and polarization vectors in holomorphic coordinates::

$$\begin{aligned} p_k^\mu &= E_k^{\text{in}} \left(1, \frac{z_k^{\text{in}} + \bar{z}_k^{\text{in}}}{1 + z_k^{\text{in}} \bar{z}_k^{\text{in}}}, \frac{-i\left(z_k^{\text{in}} - \bar{z}_k^{\text{in}}\right)}{1 + z_k^{\text{in}} \bar{z}_k^{\text{in}}}, \frac{1 - z_k^{\text{in}} \bar{z}_k^{\text{in}}}{1 + z_k^{\text{in}} \bar{z}_k^{\text{in}}}\right), \\ p_k'^\mu &= E_k^{\text{out}} \left(1, \frac{z_k^{\text{out}} + \bar{z}_k^{\text{out}}}{1 + z_k^{\text{out}} \bar{z}_k^{\text{out}}}, \frac{-i\left(z_k^{\text{out}} - \bar{z}_k^{\text{out}}\right)}{1 + z_k^{\text{out}} \bar{z}_k^{\text{out}}}, \frac{1 - z_k^{\text{out}} \bar{z}_k^{\text{out}}}{1 + z_k^{\text{out}} \bar{z}_k^{\text{out}}}\right), \\ q^\mu &= \omega \left(1, \frac{z + \bar{z}}{1 + z\bar{z}}, \frac{-i\left(z - \bar{z}\right)}{1 + z\bar{z}}, \frac{1 - z\bar{z}}{1 + z\bar{z}}\right), \\ \varepsilon^{+\mu}(q) &= \frac{1}{\sqrt{2}} \left(\bar{z}, 1, -i, -\bar{z}\right), \end{aligned} \tag{54}$$

Taking into account (50), the S-matrix element may be written as

$$\begin{aligned} \langle z_1^{\text{out}}, \cdots | : \mathcal{O}_{zz} S : | z_1^{\text{in}}, \cdots \rangle = &- \frac{\kappa}{4\pi(1 + z\bar{z})^2} \lim_{\omega \to 0^+} \Big[\omega \langle z_1^{\text{out}}, \cdots | a_+^{\text{out}}(\omega\hat{x}) S | z_1^{\text{in}}, \cdots \rangle \\ &+ \omega \langle z_1^{\text{out}}, \cdots | S a_-^{\text{in}}(\omega\hat{x})^\dagger | z_1^{\text{in}}, \cdots \rangle \Big]. \end{aligned} \tag{55}$$

Here, the fact that $a_-^{\text{out}}(\omega\hat{x})^\dagger$ ($a_+^{\text{in}}(\omega\hat{x})$) annihilates the out (in) state for $\omega \to 0$ was used. The first term corresponds to an outgoing positive helicity soft graviton, while the second – to an incoming negative helicity, both with spatial momenta $\omega\hat{x}$. Therefore, the sum of two amplitudes can be combined in one term

$$\langle z_1^{\text{out}}, \cdots | : \mathcal{O}_{zz} S : | z_1^{\text{in}}, \cdots \rangle = -\frac{\kappa}{2\pi(1 + z\bar{z})^2} \lim_{\omega \to 0^+} \left[\omega \langle z_1^{\text{out}}, \cdots | a_+^{\text{out}}(\omega\hat{x}) S | z_1^{\text{in}}, \cdots \rangle \right]. \tag{56}$$

Writing now the soft graviton theorem (30) with a positive helicity outgoing graviton

$$\begin{aligned} &\lim_{\omega \to 0} \left[\omega \langle z_1^{\text{out}}, \cdots | a_+^{\text{out}}(\vec{q}) S | z_1^{\text{in}}, \cdots \rangle \right] = \\ &= \frac{\kappa}{2} \lim_{\omega \to 0^+} \left[\sum_{k=1}^m \frac{\omega [p_k' \cdot \varepsilon^+(q)]^2}{p_k' \cdot q} - \sum_{k=1}^n \frac{\omega [p_k \cdot \varepsilon^+(q)]^2}{p_k \cdot q} \right] \langle z_1^{\text{out}}, \cdots | S | z_1^{\text{in}}, \cdots \rangle. \end{aligned} \tag{57}$$

Substituting (57) into (56) and using coordinates (54), one finds the final expression for S-matrix element with insertion of $\mathcal{O}_{zz}$:

$$\begin{aligned} \langle z_1^{\text{out}}, \cdots | : \mathcal{O}_{zz} S : | z_1^{\text{in}}, \cdots \rangle = &\frac{8G}{(1 + z\bar{z})^2} \langle z_1^{\text{out}}, \cdots | S | z_1^{\text{in}}, \cdots \rangle \\ &\times \left[\sum_{k=1}^m \frac{E_k^{\text{out}}(\bar{z} - \bar{z}_k^{\text{out}})}{(z - z_k^{\text{out}})(1 + z_k^{\text{out}}\bar{z})} - \sum_{k=1}^n \frac{E_k^{\text{in}}(\bar{z} - \bar{z}_k^{\text{in}})}{(z - z_k^{\text{in}})(1 + \bar{z}_k^{\text{in}}\bar{z})} \right]. \end{aligned} \tag{58}$$

Now, to reconnect with Ward identities, one should remember that $P_z = \frac{1}{4G}\gamma^{z\bar{z}}\partial_z\mathcal{O}_{zz}$ and calculate matrix element with P_z inserted:

$$\begin{aligned} \langle z_1^{\text{out}}, \cdots | : P_z S : | z_1^{\text{in}}, \cdots \rangle &= \frac{1}{4G} \gamma^{z\bar{z}} \partial_{\bar{z}} \langle z_1^{\text{out}}, \cdots | : \mathcal{O}_{zz} S : | z_1^{\text{in}}, \cdots \rangle \\ &= \langle z_1^{\text{out}}, \cdots | S | z_1^{\text{in}}, \cdots \rangle \left[\sum_{k=1}^m \frac{E_k^{\text{out}}}{z - z_k^{\text{out}}} - \sum_{k=1}^n \frac{E_k^{\text{in}}}{z - z_k^{\text{in}}} \right] \\ &+ \langle z_1^{\text{out}}, \cdots | S | z_1^{\text{in}}, \cdots \rangle \left[\sum_{k=1}^m \frac{E_k^{\text{out}} z_k^{\text{out}}}{1 + z_k^{\text{out}}\bar{z}} - \sum_{k=1}^n \frac{E_k^{\text{in}} z_k^{\text{in}}}{1 + z_k^{\text{in}}\bar{z}} \right]. \end{aligned} \tag{59}$$

In this expression the last term vanishes due to momentum conservation and what left is nothing else but the Ward identity

$$\langle z_1^{\text{out}}, \cdots | : P_z S : | z_1^{\text{in}}, \cdots \rangle = \langle z_1^{\text{out}}, \cdots | S | z_1^{\text{in}}, \cdots \rangle \left[\sum_{k=1}^{m} \frac{E_k^{\text{out}}}{z - z_k^{\text{out}}} - \sum_{k=1}^{n} \frac{E_k^{\text{in}}}{z - z_k^{\text{in}}} \right], \tag{60}$$

which was obtained using the machinery of quantum field theory approach.

1.1.5 Memory Effect ↔ Soft Theorem (Fourier Transform)

Recall that the Bondi news tensor is defined as the proper time derivative of the shear: $N_{zz} = \partial_u C_{zz}$. The total permanent shift in the metric is just the integral of the news over all time:

$$\Delta C_{zz} = \int_{-\infty}^{\infty} du \, \partial_u C_{zz} = \int_{-\infty}^{\infty} du \, N_{zz}(u) \,. \tag{61}$$

But this integral is exactly the zero-frequency limit of the Fourier transform of the news tensor!

$$\Delta C_{zz} = \lim_{\omega \to 0} \int_{-\infty}^{\infty} du \, e^{i\omega u} N_{zz}(u) \equiv N_{zz}^{\omega=0} \,. \tag{62}$$

Looking back at our equation (48), we see that the zero-frequency news N_{zz}^0 is exactly the zero-energy limit of the graviton annihilation operator $\lim_{\omega \to 0} \omega a^{out}(\omega)$. Thus, the classical memory shift ΔC_{zz} is mathematically equivalent to the insertion of a soft graviton.

1.1.6 Memory Effect ↔ Asymptotic Symmetry (Vacuum Transition)

To complete the IR triangle, we need to understand how does the shear C_{zz} transform under a BMS supertranslation $f(z, \bar{z})$. We should act with a Lie derivative on a corresponding component of metric tensor $\delta g_{zz} = \mathcal{L}_\xi g_{zz}$. As a result we will obtain the following constraint on the shear tensor:

$$\delta_f C_{zz} = -2D_z^2 f. \tag{63}$$

If we consider the spacetime before the wave passes ($u \to -\infty$) and after it passes ($u \to +\infty$), both regions are empty Minkowski vacua (news $N_{zz} = 0$). However, the passing wave creates a permanent shift ΔC_{zz}. This implies that the initial and final vacua are not the same; they differ precisely by a BMS supertranslation with parameter f, such that:

$$\Delta C_{zz} = -2D_z^2 f \,. \tag{64}$$

Therefore, the memory effect is a physical, measurable transition between two degenerate Minkowski vacua related by a supertranslation.

1.2 2D CFT basics

To appreciate all the beauty of celestial holography program, we need to get to know some basics of conformal field theory. This conformal field theory will be considered on a 2-dimensional celestial sphere mentioned above, and therefore, we will restrict ourselves to introduce the basics of 2D CFT.

1.2.1 Conformal transformations in D dimensions

Definition 3. *By denoting the metric as $g_{\mu\nu}(x)$ we say that a coordinate transformation $x \to x'$ is* ***conformal*** *if the metric is scale-invariant:*

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{65}$$

The word *conformal* arises from the fact that conformal transformations preserve angles between curves. To see which transformations can be conformal, we should look in the infinitesimal one $x'^\mu = x^\mu + \epsilon^\mu(x)$ and see how metric changes up to first order in ϵ:

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda}(x) = (\delta_\mu^\rho - \partial_\mu \epsilon^\rho)(\delta_\nu^\lambda - \partial_\nu \epsilon^\lambda) g_{\rho\lambda} = g_{\mu\nu} - (\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu) \tag{66}$$

where we have used that $\epsilon^\mu(x) = \epsilon^\mu(x' - \epsilon(x')) = \epsilon^\mu(x') - \epsilon^\nu(x') \partial_\nu \epsilon^\mu(x') + o(\epsilon^2)$. From here we can see that the infinitesimal transformations satisfy:

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = (1 - \Lambda(x)) g_{\mu\nu}. \tag{67}$$

Taking now the trace of this expression, we find the form of an infinitesimal function $(1-\Lambda(x)) = \frac{2}{d}\partial_\mu \epsilon^\mu \equiv f(x)$.

Now we apply one more derivative to the equation and for simplicity consider flat spacetime $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\partial_\rho \partial_\nu \epsilon_\mu + \partial_\rho \partial_\mu \epsilon_\nu = \partial_\rho f \eta_{\mu\nu}, \tag{68}$$

$$\partial_\nu \partial_\mu \epsilon_\rho + \partial_\nu \partial_\rho \epsilon_\mu = \partial_\nu f \eta_{\mu\rho}, \tag{69}$$

$$\partial_\mu \partial_\rho \epsilon_\nu + \partial_\mu \partial_\nu \epsilon_\rho = \partial_\mu f \eta_{\nu\rho}. \tag{70}$$

If we now sum (68)+(69)-(70) and then contract with $\eta^{\nu\rho}$, we arrive to an equation:

$$2\partial^2 \epsilon_\mu = (2-d)\partial_\mu f. \tag{71}$$

Taking now derivative ∂^μ from both sides and remembering $f(x) = \frac{2}{d}\partial_\mu \epsilon^\mu$, we arrive to

$$(d-1)\partial^2 f = 0. \tag{72}$$

Solving in for $d \geq 3$ we obtain the following types of conformal transformations with their infinitesimal generators:

$$\begin{aligned}
&\text{(translations)} && x'^\mu = x^\mu + a^\mu, && P_\mu - i\partial_\mu; && (73)\\
&\text{(dilations)} && x'^\mu = \alpha x^\mu, && D = -ix^\mu \partial_\mu; && (74)\\
&\text{(rigid rotations)} && x'^\mu = M^\mu_{\ \nu} x^\nu, && L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu); && (75)\\
&\text{(SCT)} && x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}, && K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu). && (76)
\end{aligned}$$

However, the most interesting case for us, namely $d = 2$, will be considered in the following.

1.2.2 Conformal transformations in 2 dimensions

The case of 2D is of special interest for us because as we have seen, the celestial sphere is 2-dimensional. Therefore, to ask for conformal symmetry on a celestial sphere, we need to reduce our search to $D = 2$.

Let's consider a plane parametrized by (z^0, z^1) and a change of coordinates $z \to w$, under which the metric tensor transforms as follows:

$$g'^{\mu\nu}(w) = \frac{\partial w^\mu}{\partial z^\alpha}\frac{\partial w^\nu}{\partial z^\beta} g^{\alpha\beta}(z). \tag{77}$$

If this coordinate is conformal, then after applying it (since the original metric is flat) the conditions (65) for the components g'^{00} and g'^{11}, g'^{01} and g'^{10} become:

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2, \tag{78}$$
$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0.$$

When solving this w.r.t. $\frac{\partial w^0}{\partial z^0}, \frac{\partial w^0}{\partial z^1}, \frac{\partial w^1}{\partial z^0}, \frac{\partial w^1}{\partial z^1}$, we obtain the Cauchy-Rieman conditions for holomorhic

$$\frac{\partial w^0}{\partial z^1} = \frac{\partial w^1}{\partial z^0}, \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}, \tag{79}$$

or antiholomoprhic functions

$$\frac{\partial w^0}{\partial z^1} = -\frac{\partial w^1}{\partial z^0}, \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1}. \tag{80}$$

This motivates us to use holomorphic and antiholomorphic coordinates $(z, \bar{z})$ from now on:

$$z = z^0 + iz^1, \tag{81}$$

$$\bar{z} = z^0 - iz^1, \tag{82}$$

$$\partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \equiv \partial, \tag{83}$$

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \equiv \bar{\partial}. \tag{84}$$

In these coordinates we rewrite the metric tensor as:

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{85}$$

In this language, conditions (78) become just $\bar{\partial} w = 0$, which means conformal transformations are just holomorphic (or analytic) functions on the complex plane $f(z)$.

Important remark here is to remember that originally (z^0, z^1) were real independent coordinates in $\mathbb{R}^2$, and therefore $(z, \bar{z})$ in principle should be considered as two dependent complex coordinates. In practice, we say that we treat z and $\bar{z}$ as two independent complex variables on $\mathbb{C}^2$, and z^0, z^1 therefore are complexified. However, when we return back to our physical space, we should imply condition $\bar{z} = z^*$, returning to our real surface.

1.2.3 Global conformal transformations

The Cauchy-Riemann conditions are local conditions, which in general can mean that $f(z)$ is not well-defined everywhere, but only in some area. Therefore, we will distinct global conformal transformations that are well-defined on the whole Riemann sphere and local ones that are not.

The global conformal transformations (a.k.a. Moebius or projective transformations)

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}, \tag{86}$$

form a $SL(2, \mathbb{C})$ group of 2x2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with unit determinant.

A composition of two Moebius transformations is indeed another one $f_2 \circ f_1$ and is given by matrix multiplication of corresponding matrices $A_1 A_2$, therefore forming a group.

1.2.4 Conformal generators

We now turn our attention to the local conformal group. Any infinitesimal holomorphic transformatino can be expressed as following:

$$z' = z + \epsilon(z), \quad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}. \tag{87}$$

Here the transformation admits a Laurent expansion around zero, but is a holomorphic function in a ring around zero. Therefore, to see which generators induce such mapping, we should expand up to first order in ϵ a scalar (for simplicity) field $\phi(z, \bar{z})$ living on the plane after an infinitesimal transformation:

$$\phi'(z', \bar{z}') = \phi(z, \bar{z}) = \phi(z', \bar{z}') - \epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z', \bar{z}'), \tag{88}$$

$$\delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi = \sum_n \left[c_n l_n \phi(z, \bar{z}) + \bar{c}_n \bar{l}_n \phi(z, \bar{z}) \right], \tag{89}$$

where $l_n, \bar{l}_n$ are nothing else but generators of conformal algebra

$$l_n = -z^{n+1}\partial_z, \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{90}$$

obeying commutation relations:

$$[l_m, l_n] = (m - n) l_{m+n}, \tag{91}$$

$$[\bar{l}_m, \bar{l}_n] = (m - n) \bar{l}_{m+n}, \tag{92}$$

$$[l_m . \bar{l}_n] = 0. \tag{93}$$

Both l_n and $\bar{l}_n$ obey Witt algebra, so sometimes we will say that algebra of conformal generators is nothing else but two copies of Witt algebra: Witt$\otimes$Witt.

One may notice that generators l_{-1}, l_0, l_1 form subalgebra which reproduces the global conformal algebra $sl(2, \mathbb{C})$. Notice also that when $n = -1, 0, 1$, the generators are holomorphic on the whole Riemann sphere.

Exercise 3. *Demanding that the generators should be holomorphic at $z = 0$ and $z = \infty$ (make an inversion $z = \lim_{w \to 0} \frac{1}{w}$) show that $n \in \{-1, 0, 1\}$.*

For some intuition we will list some of the generators and their action below:

- $l_{-1} = -\partial_z$ – translations on the plane;
- $l_0 = -z\partial_z$ – scale transformations + rotations on the plane;
- $l_1 = -z^2\partial_z$ – SCT (special conformal transformations) on the plane;
- $l_n + \bar{l}_n, i(l_n - \bar{l}_n)$ – generators that preserve the real surface $z^0, z^1 \in \mathbb{R}$.

Here I specified that all these transformations are "on the plane" so nobody will try to directly connect translations in 4D bulk with these ones. Actions of aforementioned symmetries are considered and called translations (or rotations, dilatations, etc) only on the 2D plane.

1.2.5 Primary fields

To build a space of states in 2D CFT, we need a specific set of local fields called primary fields. These fields are annihilated by lowering generators of conformal algebra and have a special transformation rule they obey. All the other fields in CFT can be obtained by acting with raising generators on the primary fields and are called their descendants.

Before defining a primary field, we should introduce some important notation:

Definition 4. *For a field with scaling dimension Δ and spin s we define $h = \frac{1}{2}(\Delta + s)$ to be his* ***holomorphic conformal dimension****, while $\bar{h} = \frac{1}{2}(\Delta - s)$ an* ***antiholomorphic conformal dimension****.*

And now we can define the primary fields:

Definition 5. *A field that under a conformal map $z \to w(z), \bar{z} \to \bar{w}(\bar{z})$ transforms as*

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z})$$

is called a ***quasi-primary field****.*

Notice that a quasi-primary field can transform this way under global $SL(2,\mathbb{C})$ transformations only. When a field transforms according to this rule under *any* local conformal transformation, we call it a ***primary*** field.

Rigorosity Alert 1. *Now we should make an important note: even though it would be mathematically correct, in the latter we will not use the word "quasi-primary", when talking about fields transforming this way under global $SL(2,\mathbb{C})$ transformations. We will start calling these fields primaries, and later on we will discover that they are primaries indeed if we consider superrotation symmetry.*

1.2.6 Correlation functions

We should also quickly mention which symmetries and properties do correlation functions have in CFT.

We start with a two-point function for a quasi-primary field with action $S[\phi]$ (using complex coordinates $(z_i, \bar{z}_i)$):

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{1}{Z}\int [d\Phi]\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\exp\{-S[\Phi]\}. \tag{94}$$

The conformal invariance of the action and of the functional integration measure leads to the following transformation of the correlation function $z \to w(z), \bar{z} \to \bar{w}(\bar{z})$:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \left(\frac{dw}{dz}\right)^{h_1}_{z_1} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}_1}_{\bar{z}_1} \left(\frac{dw}{dz}\right)^{h_2}_{z_2} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{h}_2}_{\bar{z}_2} \langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)\rangle. \tag{95}$$

If we specialize to a scale and rotation transformation $z \to \lambda z$, $\bar{z} \to \bar{\lambda}\bar{z}$ we obtain

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \lambda^{h_1+h_2}\bar{\lambda}^{\bar{h}_1+\bar{h}_2}\langle \phi_1(\lambda z_1, \bar{\lambda}\bar{z}_1)\phi_2(\lambda z_2, \bar{\lambda}\bar{z}_2)\rangle. \tag{96}$$

Meanwhile translation invariance requires that the correlator depends only on the differences $z_{12} = z_1 - z_2$ and $\bar{z}_{12} = \bar{z}_1 - \bar{z}_2$. Taking into account the scaling and rotation invariance, we get:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{z_{12}^{h_1+h_2}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}, \tag{97}$$

where C_{12} is a constant coefficient. It only remains to use the invariance under special conformal transformations. We recall that, for such a transformation $w(z) = \frac{z}{1-bz}$, the derivative is:

$$\frac{dw}{dz} = \frac{1}{(1-bz)^2}, \quad \frac{d\bar{w}}{d\bar{z}} = \frac{1}{(1-\bar{b}\bar{z})^2}. \tag{98}$$

The transformation for the distance $w_{12} = w_1 - w_2$ is given by:

$$w_{12} = \frac{z_{12}}{(1-bz_1)(1-bz_2)}, \quad \bar{w}_{12} = \frac{\bar{z}_{12}}{(1-\bar{b}\bar{z}_1)(1-\bar{b}\bar{z}_2)}, \tag{99}$$

therefore the covariance of the correlation function implies

$$\frac{C_{12}}{z_{12}^{h_1+h_2}\,\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}} = \frac{C_{12}}{\gamma_1^{h_1}\bar{\gamma}_1^{\bar{h}_1}\gamma_2^{h_2}\bar{\gamma}_2^{\bar{h}_2}}\frac{(\gamma_1\gamma_2)^{h_1+h_2}(\bar{\gamma}_1\bar{\gamma}_2)^{\bar{h}_1+\bar{h}_2}}{z_{12}^{h_1+h_2}\,\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}, \tag{100}$$

with

$$\gamma_i = (1-bz_i), \quad \bar{\gamma}_i = (1-\bar{b}\bar{z}_i). \tag{101}$$

This constraint is identically satisfied only if $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$. In other words, two quasi-primary fields are correlated only if they have the same conformal weights:

$$\langle\phi_1\phi_2\rangle = \begin{cases} \dfrac{C_{12}}{z_{12}^{2h}\,\bar{z}_{12}^{2\bar{h}}} & \text{if } h_1 = h_2 = h \text{ and } \bar{h}_1 = \bar{h}_2 = \bar{h} \\ 0 & \text{otherwise} \end{cases}. \tag{102}$$

Performing similar analysis, denoting $z_{ij} = z_i - z_j$ and $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$, we can obtain the expression for a 3-point function:

$$\langle\phi_1\phi_2\phi_3\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\,z_{23}^{h_2+h_3-h_1}\,z_{13}^{h_3+h_1-h_2}} \times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\,\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\,\bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \tag{103}$$

Exercise 4. *Perform a similar analysis and obtain 3-point function.*

Unfortunately, conformal covariance can only carry us so far. When considering the 4-point function, we discover that the form of it might depend arbitarily on cross-ratios. There is a finite number of linearly independent cross-ratios that one can construct in 2D CFT:

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad 1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \qquad \frac{\eta}{1-\eta} = \frac{z_{12}z_{34}}{z_{14}z_{23}}, \tag{104}$$

but unfortunately an infinite number of real functions $f(\eta, \bar{\eta})$ one can construct out of them:

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = f(\eta,\bar{\eta})\prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j}\,\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}. \tag{105}$$

As we can see, the 4-point function is no longer uniquely fixed by conformal invariance.

1.2.7 Ward identities for correlation functions

Since we have symmetry and we have correlation functions of fields satisfying this symmetry, it is reasonable to ask about Ward identities of the theory. For our purposes, we will restrict ourselves to just postulating them.

For correlator $\langle X\rangle$ of n primary fields, we have two equivalent formulations of Ward identity (which we will not derive, but interested reader can find derivation in detail in [20]):

$$\begin{gathered} \delta_\epsilon\langle X\rangle = -\sum_i\left[\epsilon(w_i)\partial_{w_i} + \partial\epsilon(w_i)h_i\right]\langle X\rangle \\ \Updownarrow \quad \text{absolutely the same thing!} \\ \langle T(z)X\rangle = \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i} + \frac{h_i}{(z-w_i)^2}\right]\langle X\rangle + \text{reg.} \end{gathered} \tag{106}$$

where 2D momentum-energy tensor is $T(z) = -2\pi T_{zz}$, $\bar{T}(\bar{z}) = -2\pi T_{\bar{z}\bar{z}}$ and

For the global conformal transformations generated by $\{l_{\pm 1}, l_0\}$, this variation must vanish, and therefore exist the following three relations on correlators of primary fields, which are global Ward identities:

$$
\begin{aligned}
\sum_i \partial_{w_i} \langle X \rangle &= 0 \\
\sum_i (w_i \partial_{w_i} + h_i) \langle X \rangle &= 0 \\
\sum_i (w_i^2 \partial_{w_i} + 2 w_i h_i) \langle X \rangle &= 0
\end{aligned}
\tag{107}
$$

1.2.8 Operator Product Expansion

When the position of two or more fields coincide, it is usual to have singularities in correlation functions. The operator product expansion, or OPE, is a representation of a product of operators at positions z and w, by a sum of terms, each being a [well defined as $z \to w$ operator] $\times$ [possibly divirging as $z \to w$ function of $(z-w)$].

Definition 6. *OPE of two fields is an expansion*

$$
A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}
\tag{108}
$$

as $z \to w$, where fields $\{AB\}_n(w)$ are nonsingular at $w = z$.

The most important part of the OPE is the singular part. Therefore, we will usually disregrad the regular in $(z-w)$ terms and will put the symbol when we mean equality modulo expressions regular as $z \to w$. For example, OPE of a primary field ϕ of conformal dimensions h and $\bar{h}$:

$$
T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \phi(w,\bar{w}),
\tag{109}
$$

$$
\bar{T}(\bar{z})\phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} \phi(w,\bar{w}).
\tag{110}
$$

1.2.9 Closing remarks on CFT

Conformal field theory is a fascinating and deep area that any theoretical physicist needs to explore sooner or later. It has applications in surprisingly many domains of theoretical physics at different levels of abstraction. String theory and the AdS/CFT correspondence (as follows from the name) simply could not exist without conformal field theory.

However, in these lectures we will limit ourselves to the essentials of CFT mentioned above, hoping that they will be enough to set an introductory bridge to celestial holography.

Rigorosity Alert 2. *If between any lines that I wrote you start asking yourself the question "why?" I strongly recommend opening the Big Yellow Book [20] or a smaller blue book [21].*

2 Lecture 2. From momentum space to celestial sphere

In order to establish holographic correspondence, we need a dictionary to translate the physics in the 4-dimensional bulk to the physics on the 2-dimensional celestial sphere. Moreover, we need to understand how conformal algebra is connected with algebra of Lorentz generators, how to construct the basis wavefunctions and which properties they possess. With this vocabulary established, we can define precisely what a celestial amplitude is and study how it transforms.

2.1 Lorentz algebra and $sl(2,\mathbb{C})$ algebra

To explicitly see the correspondence between generators of these two algebras, we start from the Lorentz generators that in general case look like

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \tag{111}$$

where $L_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu)$ is orbital angular momentum and $S_{\mu\nu}$ is spin generator. Right now we will concentrate on the scalar case $S^{\mu\nu} = 0$ for simplicity since our course has introductory character. Spinning case is considered, for example, in [22].

The orbital part consists of rotations J_i and boosts K_j:

$$J_i = -\varepsilon_{ijk} x^j \partial_k, \qquad i, j, k \in \{1, 2, 3\} \tag{112}$$

$$K_i = -(x^0 \partial_i + x^i \partial_0). \tag{113}$$

They obey the standard Lorentz algebra:

$$\begin{aligned} [J_i, J_j] &= \varepsilon_{ijk} J_k, \\ [K_i, K_j] &= -\varepsilon_{ijk} J_k, \\ [J_i, K_j] &= \varepsilon_{ijk} K_k. \end{aligned} \tag{114}$$

Exercise 5. *Check that (114) holds.*

Rigorosity Alert 3. *There exist several ways of showing equivalence between action of Lorentz group and $SL(2,\mathbb{C})$ group on momentum space. I will demonstrate it using the so-called Milne slicing, when we start by parametrizing massive states in Minkowski near $i^\pm$, but then take one of the parameters to infinity, obtaining massless limit. It can be also obtained in a possibly clearer way if one uses the spinor helicity notation, but I will not overload this short lecture course with introducing it. For the ones who crave spinor helicity formalism and action of $SL(2,\mathbb{C})$ on spinors, I refer to a review [23].*

In hyperbolic coordinates $(\tau, \rho, z, \bar{z})$ the Minkowski metric takes the form

$$ds^2 = -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{\rho^2} + \rho^2 dz d\bar{z} \right). \tag{115}$$

These are related to the usual Cartesian coordinates

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \tag{116}$$

by

$$\tau = \sqrt{(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2} \tag{117}$$

$$z = \frac{X^1 + iX^2}{X^0 + X^3} \tag{118}$$

$$\rho = \frac{X^0 + X^3}{\sqrt{(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2}}, \tag{119}$$

with inverse

$$X^0 = \frac{1}{2}\tau\rho(1 + z\bar{z} + \rho^{-2}) \tag{120}$$

$$X^1 = \frac{1}{2}\tau\rho(z + \bar{z}) \tag{121}$$

$$X^2 = -\frac{i}{2}\tau\rho(z - \bar{z}) \tag{122}$$

$$X^3 = \frac{1}{2}\tau\rho(1 - z\bar{z} - \rho^{-2}). \tag{123}$$

The hyperbolic coordinates represent Minkowski spacetime as a foliation (labeled τ) of 3D constant curvature hyperbolic spaces. We label spacelike slices in the future (past) lightcone of the origin by $\tau > 0$ ($\tau < 0$). We are especially interested in the $\pm\tau \to \infty$ slices that approach $\mathscr{I}^{\pm}$. We denote them by $H_3^{\pm}$. The de Sitter slices at spacelike separations from the origin are labeled by positive imaginary τ. The asymptotic slice $\tau \to i\infty$ is denoted by dS_3^0. This is illustrated in Figure 4. The $\rho = \infty$ boundary of H_3^+ will be referred to as the "future celestial sphere" and denoted $\mathcal{CS}^+$. The analogously defined past celestial sphere will be denoted $\mathcal{CS}^-$.

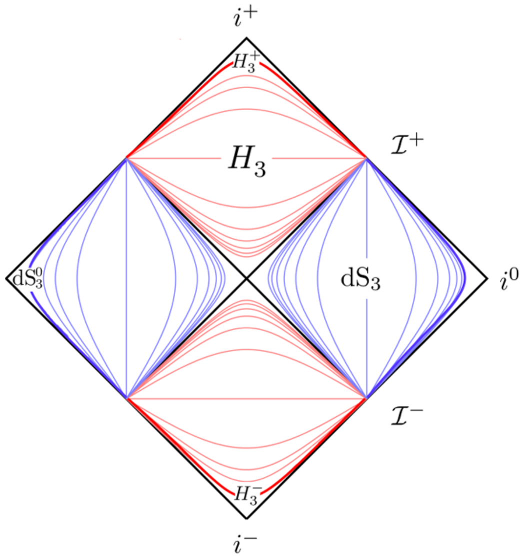


Figure 4: Penrose diagram of hyperbolic slicing of Minkowski space. The slices correspond to surfaces of constant τ. The slices in the past and future lightcones of the origin have the geometry of H_3, and the slices with spacelike separation from the origin have the geometry of dS_3.

As we can see from the picture, the hyperbolic slicing can only take us close to the upper boundary of future celestial sphere $\mathscr{I}^+_+$ that is located "near" i^+ on the Penrose diagram or to the lower boundary of past celestial sphere $\mathscr{I}^-_-$ that is located "near" i^-. However, the form of Lorentz generators does not depend on the exact S^2-slice of null infinity and is preserved in the same form for both $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$.

In these coordinates, Lorentz generators (114) take the following form:

$$J_1 = -\frac{i}{2}\left[(z^2-1)\partial_z - (\bar{z}^2-1)\partial_{\bar{z}}\right], \qquad K_1 = -\frac{1}{2}\left[(z^2-1)\partial_z + (\bar{z}^2-1)\partial_{\bar{z}}\right], \tag{124}$$

$$J_2 = -\frac{1}{2}\left[(z^2+1)\partial_z + (\bar{z}^2+1)\partial_{\bar{z}}\right], \qquad K_2 = \frac{i}{2}\left[(z^2+1)\partial_z - (\bar{z}^2+1)\partial_{\bar{z}}\right], \tag{125}$$

$$J_3 = -i(z\partial_z - \bar{z}\partial_{\bar{z}}), \qquad K_3 = -(z\partial_z + \bar{z}\partial_{\bar{z}}). \tag{126}$$

Exercise 6. *Derive the form of Lorentz generators (114) in coordinates $(\tau, \eta, \theta, \varphi)$, then take the limit $\eta \to \infty$ and go from spherical coordinates (θ, φ) to $(z, \bar{z})$*

The combinations of derivatives and powers of z give us a hint that conformal $sl(2, \mathbb{C})$ generators are hidden inside the Lorentz ones. Indeed, if we now we construct linear combinations from these generators as follows:

$$L_0 = -\frac{i}{2}(J_3 + iK_3), \qquad \bar{L}_0 = \frac{i}{2}(J_3 - iK_3) \tag{127}$$
$$L_1 = -\frac{i}{2}(J_1 + iK_1 + i(J_2 + iK_2)), \qquad \bar{L}_1 = \frac{i}{2}(J_1 - iK_1 - i(J_2 - iK_2)), \tag{128}$$
$$L_{-1} = \frac{i}{2}(J_1 + iK_1 - i(J_2 + iK_2)), \qquad \bar{L}_{-1} = -\frac{i}{2}(J_1 - iK_1 + i(J_2 - iK_2)), \tag{129}$$

they will obey the commutation relations of $sl(2,\mathbb{C})$ algebra.

Exercise 7. *Explicitly check that $sl(2,\mathbb{C})$ algebra is obeyed.*

2.1.1 The Millennium Falcon and light aberration

Time to return to the cover of our lecture notes, depicting Han Solo and Chewbacca enjoying a hyper jump. What would they actually see? How would the celestial sphere they see actually transform when they move at relativistic speeds?

According to special relativity when an observer starts moving with speeds close to the speed of light, the angle from the source that at resting frame was θ will transform to θ':

$$\tan\left(\frac{\theta'}{2}\right) = \sqrt{\frac{1-v}{1+v}}\tan\left(\frac{\theta}{2}\right). \tag{130}$$

When we rewrite it in stereographic coordinates $z = \tan\left(\frac{\theta}{2}\right)e^{i\phi}$, $\bar{z} = \tan\left(\frac{\theta}{2}\right)e^{-i\phi}$:

$$z' = \tan\left(\frac{\theta'}{2}\right)e^{i\phi} = \sqrt{\frac{1-v}{1+v}}\tan\left(\frac{\theta}{2}\right)e^{i\phi} = \sqrt{\frac{1-v}{1+v}}z, \tag{131}$$

and analogously $\bar{z}' = \sqrt{\frac{1-v}{1+v}}\bar{z}$, which correponds to dilatations on a celestial sphere. Notice that the scaling factor $\sqrt{\frac{1-v}{1+v}} < 1$, which means we are moving towards the north pole $(z,\bar{z}) = (0,0)$ in the direction of movement. This can be depicted as Han Solo and Chewbacca seeing distant stars, even the ones behind them being "collected" near their direction of movement. As we can see from the relation between $so(1,3)$ and $sl(2,\mathbb{C})$ algebra, this shifting of stars on a sphere will be purely conformal transformation, preserving angles.

2.2 Conformal primary wavefunctions

2.2.1 Derivation

Conformal primary wavefunctions Ψ should satisfy two conditions (again, we consider the scalar case):

1. Highest weight condition, since they are *conformal* and *primary*:
$$L_1\Psi = \bar{L}_1\Psi = 0; \tag{132}$$
2. Wave equation, since they are *wave* functions:
$$(\nabla^2 - m^2)\Psi = 0. \tag{133}$$

Now we notice that function

$$\Psi_\Delta \sim \frac{1}{(x^0 + x^3)^\Delta} \tag{134}$$

satisfies the first condition:

$$L_1\Psi_\Delta = \bar{L}_1\Psi_\Delta = 0, \tag{135}$$

while also being an eigenfunction of the Lorentz boosts in the x^3 direction:

$$\begin{aligned}(L_0 + \bar{L}_0)\Psi_\Delta &= \Delta\Psi_\Delta, \\ (L_0 - \bar{L}_0)\Psi_\Delta &= 0.\end{aligned} \tag{136}$$

If we multiply Ψ_Δ by an arbitary function $f(x^\mu x_\mu)$, it will still satisfy the conditions (135), (136). Also notice that sum $(x^0 + x^3)$ is given by a product of 4-vector x^μ with another 4-vector $q^\mu = (1,0,0,1)$ pointing in the direction of the coordinate x^3. Collecting everything together, we have a wavefunction looking as follows:

$$\Psi_\Delta = \frac{f(x^\mu x_\mu)}{(x\cdot q)^\Delta}. \tag{137}$$

Exercise 8. *Apply the $SL(2,\mathbb{C})$ generators and see that (135)-(136) actually holds for chosen wavefunction.*

Now we should also remember that it should as well obey wave equation. In the rest of this section we will denote $x^2 = x^\mu x_\mu$ for simplicity. Not to confuse this notation with the 2nd component of 4-vector x^μ. So, the wave equation for a function of such form is given by:

$$4x^2 f''(x^2) - 4(2\Delta - 2)f'(x^2) - m^2 f(x^2) = 0. \tag{138}$$

Exercise 9. *Show that (133) indeed yields this differential equation.*

Bringing this equation to a Bessel equation, we first make a change of variable $x^2 \to y = \sqrt{x^2}$, and it transforms to:

$$y^2 f''_y - (2\Delta - 3)y f'_y - m^2 y^2 f = 0. \tag{139}$$

We will search solution in a form $f(y) = y^\alpha g(y)$, and substituting this ansatz we fix $\alpha = \Delta - 1$ for the above equation to fit the canonical Bessel equation. The equation will now take form:

$$y^2 g''(y) + y g'(y) - \big[m^2 y^2 + (\Delta - 1)^2\big] g(y) = 0. \tag{140}$$

The last step now is to perform a scaling $z = my$, and we arrive to the known modified Bessel's equation, which solutions are linear combinations of modified Bessel functions of first and second kind. The condition that our wave function should vanish on spatial infinity $x^2 \to \infty$ picks only the wave function of second kind $K_{\Delta-1}$, arriving to the following expression for function $f(\sqrt{x^2})$:

$$f(x^2) \sim (\sqrt{x^2})^{\Delta-1} K_{\Delta-1}(m\sqrt{x^2}). \tag{141}$$

Taking it into account, up to some normalization our wavefunction will have form:

$$\Psi_\Delta(x, \vec{z}) \sim \frac{(\sqrt{x^2})^{\Delta-1}}{(q \cdot x)^\Delta} K_{\Delta-1}(m\sqrt{x^2}). \tag{142}$$

Under Lorentz transformations, q and x transform linearly:

$$q^\mu(z', \bar{z}') = \left|\frac{\partial \vec{z}'}{\partial \vec{z}}\right|^{1/2} \Lambda^\mu_\nu q^\nu(z, \bar{z}), \qquad x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{143}$$

and independently of the normalization, the wavefunction therefore will transform as a primary field:

$$\Psi_\Delta(x', \vec{z}') = \left|\frac{\partial \vec{z}'}{\partial \vec{z}}\right|^{-\Delta/2} \Psi_\Delta(x, \vec{z}). \tag{144}$$

2.2.2 Integral representation

These freshly derived wavefunctions admit the Fourier expansion on the plane waves [24]

$$\Psi_\Delta(x, \vec{z}) = \int_{\mathbb{H}_3} [d\hat{p}] G_\Delta(\hat{p}, \vec{z}) e^{im\hat{p}\cdot x}, \tag{145}$$

where the hyperbolic $SO(1,3)$-invariant measure $[d\hat{p}]$ is given by

$$\int_{\mathbb{H}_3} [d\hat{p}] \equiv \int_0^\infty \frac{dy}{y^3} \int d^2\vec{w}, \tag{146}$$

$p^\mu = m\hat{p}^\mu$ parametrized as

$$\hat{p}(y, \vec{w}) = (1 + y^2 + w\bar{w}, w + \bar{w}, -i(w - \bar{w}), 1 - y^2 - w\bar{w}), \tag{147}$$

and the $G_\Delta(\hat{p}, \vec{z})$ being the bulk-to-boundary propagator in AdS_3 [25]:

$$G_\Delta(\hat{p}, \vec{z}) = \left(\frac{y}{y^2 + |\vec{z} - \vec{w}|^2}\right)^\Delta = \frac{1}{(-\hat{p} \cdot q)^\Delta}. \tag{148}$$

From the last equality it is clear that the propagator transforms as a primary itself under an $SO(1,3)$ transformation:

$$G_\Delta(\Lambda p, \Lambda q) = \left|\frac{\partial \vec{z}'}{\partial \vec{z}}\right|^{-\Delta/2} G_\Delta(p, q). \tag{149}$$

Hyperbolic momentum parametrization

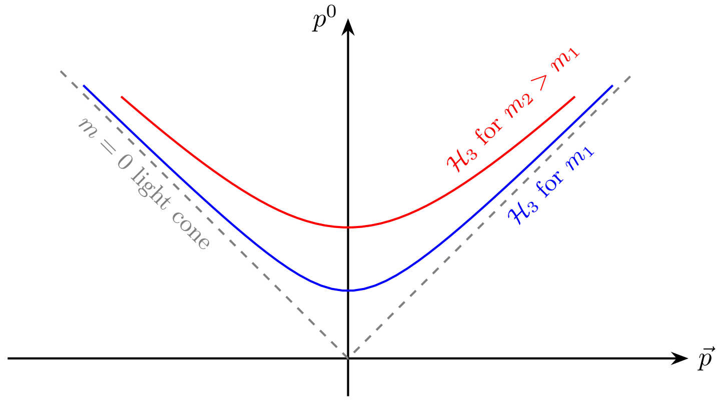


Figure 5: Mass shells $p^2 = -m^2$ in 4D Minkowski space. Limit $m \to 0$ drags the hyperboloid to the lightcone.

2.2.3 Massless case

In the beginning of this section we imposed a massive Klein-Gordon equation for the conformal primary wavefunction to obey, and during solving the differential equations the mass inevitably appeared. Now we ask ourselves: what about the massless particles? Do the massless wavefunctions change compared to the massive ones? The quick answer is yes, and their integral representations become simpler, namely they become Mellin transforms of the plane waves. The expanded explanation of this we will obtain below.

Looking closely at the hyperbolic parametrization of momenta above, we see that massless limit is obtained by sending $y \to 0$ while $\omega \equiv \frac{m}{2y}$ is held fixed. Starting with the expression for the bulk-to-boundary propagator, we consider two cases:

1. $\vec{w} \neq \vec{z}$, the bulk coordinate $\vec{w}$ doesn't lie on the boundary. In this case the limit is taken trivially:

$$\lim_{y\to 0} G_\Delta(y, \vec{w}; \vec{z}) = \lim_{y\to 0} \left(\frac{y}{y^2 + |\vec{z} - \vec{w}|^2} \right)^\Delta = \frac{y^\Delta}{|\vec{z} - \vec{w}|^{2\Delta}}; \tag{150}$$

2. $\vec{w} = \vec{z}$, the bulk coordinate $\vec{w}$ lies on the boundary. In this case such limit diverges, indicating the presence of delta function $\delta^{(2)}(\vec{z} - \vec{w})$. We take our ansatz to be $G_\Delta(y, \vec{w}; \vec{z}) = C(y) \cdot \delta^{(2)}(\vec{z} - \vec{w})$ and integrate it over the boundary $\int d^2\vec{z}$:

$$C(y) = \int d^2 z \left(\frac{y}{y^2 + |z - w|^2} \right)^\Delta = \int_0^{2\pi} d\phi \int_0^\infty r\, dr \left(\frac{y}{y^2 + r^2} \right)^\Delta = 2\pi y^\Delta \int_0^\infty \frac{r\, dr}{(y^2 + r^2)^\Delta} = \frac{\pi}{\Delta - 1} y^{2-\Delta}, \tag{151}$$

where we went to polar coordinates $z - w = re^{i\phi}$ with $d^2\vec{z} = r\, dr\, d\phi$.

Therefore, the final answer for the limimt of our propagator consists of two terms:

$$G_\Delta(y, \vec{w}; \vec{z}) = \frac{\pi}{\Delta - 1} y^{2-\Delta} \delta^{(2)}(\vec{z} - \vec{w}) + \frac{y^\Delta}{|\vec{z} - \vec{w}|^{2\Delta}} + \cdots \tag{152}$$

The first term, upon taking the integral transform (87) on it, changing variables $\int_0^\infty dy = \int_0^\infty \frac{m}{2\omega^2} d\omega$,after integrating over the bulk variable $\vec{w}$ in delta function, takes the form of a Mellin transform:

$$\psi_\Delta(x, \vec{z}) = \frac{\pi m^{-\Delta}}{2^{-\Delta}(\Delta - 1)} \int_0^\infty d\omega\, \omega^{\Delta-1} e^{i\omega \hat{q}(\vec{z})\cdot x}. \tag{153}$$

To not carry around the factors in front, we will define the normalization of massless wavefunction as following:

$$\phi_\Delta(x, \vec{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1} e^{i\omega \hat{q}(\vec{z})\cdot x} \sim \frac{1}{(\hat{q} \cdot x)^\Delta}. \tag{154}$$

2.3 Celestial basis for scattering amplitudes with examples

Using these freshly derived dictionary we can relate momentum space scattering amplitudes $\mathcal{A}$ to scattering amplitudes $\widetilde{\mathcal{A}}$ in a conformal primary basis.

Definition 7. *The* ***celestial amplitude*** *is defined to be the following transformation of momentum space amplitude and corresponds to a correlator of local primary fields $O_{\Delta_i}(z_i, \bar{z}_i)$ living on CCFT:*

$$\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = \langle O_{\Delta_1}(z_1, \bar{z}_1) \dots O_{\Delta_n}(z_n, \bar{z}_n) \rangle \equiv \prod_{i=1}^{n} \int_{\mathbb{H}_3} \frac{d^3\hat{p}_i}{p_i^0} G_{\Delta_i}(\hat{p}_i; z_i, \bar{z}_i) \mathcal{A}(\epsilon_i m_i \hat{p}_i), \tag{155}$$

where $\epsilon_i = \pm 1$ depending on whether the i^{th} particle is incoming or outgoing.

The celestial amplitude transforms as a primary field under $SL(2, \mathbb{C})$ transformation, since its building bricks, conformal primary wavefunctions, transform as primaries as well:

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{z}\,'_i(\vec{z}_i)) = \prod_{i=1}^{n} \left| \frac{\partial \vec{z}\,'_i}{\partial \vec{z}_i} \right|^{-\Delta_i/2} \widetilde{\mathcal{A}}(\Delta_i, \vec{z}_i). \tag{156}$$

Important note to remember is that conformal primary wavefunctions form a basis of solutions not for any arbitary weight Δ, but for a certain set of them. For example,

1. $\Delta = 1 + i\lambda$, $\lambda \geq 0$ for massive case;
2. $\Delta = 1 + i\lambda$, $\lambda \in \mathbb{R}$ for massless case.

Definition 8. *The range $1+i\mathbb{R}$ is called* ***principal continuous series*** *of the irreducible unitary representations of $SO(1,3)$.*

This condition can be obtained by imposing ortonormality, linear independence on the wavefunctions while also demanding that they should be delta-function-normalizable w.r.t. the Klein-Gordon inner product. How to obtain it in detailes is discussed in [24].

2.3.1 Example of 2 massless and 1 massive scalars at tree-level

It is important to see at least once how this integral transforms are being made and to obtain the direct form of some celestial amplitude. With this purpose we consider the simple example of a tree-level celestial amplitude for two massless and one massive scalars [26]. This will be also a pedagogical example, since the form of 3-point correlation function in 2D CFT is still fixed from conformal symmetry up to a coupling constant:

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \tag{157}$$

$$\times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}. \tag{158}$$

Thus, we have an expectation of what we should obtain in the end and can compare the final answer with this expectation.

We start with the momentum space 3-point interaction:

$$\mathcal{A}(\hat{p}_i) = g\delta^{(4)}(\omega_1 \hat{q}_1 + \omega_2 \hat{q}_2 - m\hat{p}). \tag{159}$$

The associated celestial amplitude is then

$$\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = g \prod_{i=1}^{2} \left(\int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} \right) \int_0^\infty \frac{dy}{y^3} \int d^2w \left(\frac{y}{y^2 + |z_3 - w|^2} \right)^{\Delta_3} \tag{160}$$

$$\times\, \delta^{(4)}(\omega_1 \hat{q}_1 + \omega_2 \hat{q}_2 - m\hat{p}). \tag{161}$$

The most tricky part when computing celestial amplitude is finding the roots of delta function. While more or less obvious in the massless case and 3-point interaction, one might find themselves in a pickle when going to more legs and considering more massive particles. However, considering deadly difficult integrals is out of the scope of our course, because this would not be welcoming at all to the listener.

We will be using the following parameterizations of momenta:

$$\begin{aligned}\hat{q} &= (1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z})\,, \\ \hat{p} &= \frac{1}{2y}\left(1+y^2+w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1-y^2-w\bar{w}\right)\end{aligned} \tag{162}$$

In these variables, we solve the momentum-conserving delta function and find that on its support,

$$\omega_2 = \frac{m^2}{4\omega_1|z_{12}|^2}, \qquad y = \frac{2m\omega_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}, \tag{163}$$

$$w = \frac{m^2 z_2 + 4\omega_1^2 z_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}, \qquad \bar{w} = \frac{m^2 \bar{z}_2 + 4\omega_1^2 \bar{z}_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}. \tag{164}$$

The Jacobian for the transformation from $(\omega_i\hat{q}_i, m\hat{p})$ to $(\omega_2, y, w, \bar{w})$ is

$$|J| = \frac{m^3(y^2+|w-z_2|^2)}{2y^4}. \tag{165}$$

We then find that

$$\frac{1}{y^3}\frac{1}{|J|}\left(\frac{y}{y^2+|w-z_3|^2}\right)^{\Delta_3}\delta\left(\omega_2-\frac{m^2}{4\omega_1|z_{12}|^2}\right)\delta\left(y-\frac{2m\omega_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}\right) \tag{166}$$

$$\times\,\delta^{(2)}\left(w-\frac{m^2 z_2+4\omega_1^2 z_1|z_{12}|^2}{m^2+4\omega_1^2|z_{12}|^2}\right) = \frac{2}{m^3}\frac{m}{2\omega_1|z_{12}|^2}\left(\frac{2m\omega_1|z_{12}|^2}{4\omega_1^2|z_{12}|^2|z_{13}|^2+m^2|z_{23}|^2}\right)^{\Delta_3}\times\delta^{(4)}. \tag{167}$$

The integrals over $y, w, \bar{w}$ are now trivial and the celestial 3-point amplitude becomes

$$\begin{aligned}\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) &= g\left(\frac{m^2}{4|z_{12}|^2}\right)^{\Delta_2-1}\frac{(2m|z_{12}|^2)^{\Delta_3}}{m^2|z_{12}|^2}\int_0^\infty d\omega_1\,\frac{\omega_1^{\Delta_1-\Delta_2+\Delta_3-1}}{(4\omega_1^2|z_{12}|^2|z_{13}|^2+m^2|z_{23}|^2)^{\Delta_3}} &(168)\\ &= \frac{gm^{2\Delta_2+\Delta_3-4}}{2^{2\Delta_2-\Delta_3-2}|z_{12}|^{2\Delta_2-2\Delta_3}}\int_0^\infty d\omega_1\,\frac{\omega_1^{\Delta_1-\Delta_2+\Delta_3-1}}{(4\omega_1^2|z_{12}|^2|z_{13}|^2+m^2|z_{23}|^2)^{\Delta_3}}, &(169)\end{aligned}$$

Upon a change of variables, the remaining integral becomes proportional to the beta function:

$$\int_0^1 dt\, t^{\alpha-1}(1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \equiv B(\alpha,\beta), \tag{170}$$

and we conclude

$$\begin{aligned}\widetilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) &= \frac{C(\Delta_1,\Delta_2,\Delta_3)}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3}|z_{13}|^{\Delta_1+\Delta_3-\Delta_2}|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}}, \\ C(\Delta_1,\Delta_2,\Delta_3) &= g\frac{m^{\Delta_1+\Delta_2-4}}{2^{\Delta_1+\Delta_2-1}}B\left(\frac{\Delta_{12}+\Delta_3}{2}, \frac{-\Delta_{12}+\Delta_3}{2}\right).\end{aligned} \tag{171}$$

Notice how the structure reproduces 3-point correlation function (158) of 2D CFT, taking into account that in the scalar case $h_i = \bar{h}_i = \frac{1}{2}\Delta_i$.

3 Lecture 3. Symmetries on the celestial sphere

3.1 Action of 4D bulk symmetries on the Celestial sphere

As was mentioned in the very beginning, holographic correspondence means that certain symmetries in the bulk are related to certain symmetries on the boundary of a spacetime. In the last lectures we saw that Lorentz transformations in the bulk of our asymptotically flat spacetime is corresponding to the global conformal $SL(2,\mathbb{C})$ transformations on the celestial sphere, and that our conformal primary basis is set of Lorentz boost eigenstates. But what about other symmetries of our spacetime? Starting with 4D translations, in this lecture we will set the symmetry correspondence between asymptotically flat spacetime and its boundary.

3.1.1 Poincaré symmetry on $\mathcal{CS}$

It is important to discuss how the Poincaré symmetry acts on celestial amplitudes. As shown in [27] the Lorentz generators act on the stripped celestial amplitudes as

$$\begin{aligned} L_0 &= 2(z\partial_z + h), \quad L_- = \partial_z, \quad L_+ = z^2\partial_z + 2zh, \\ \bar{L}_0 &= 2(\bar{z}\partial_{\bar{z}} + \bar{h}), \quad \bar{L}_- = \partial_{\bar{z}}, \quad \bar{L}_+ = \bar{z}^2\partial_{\bar{z}} + 2\bar{z}\bar{h}. \end{aligned} \tag{172}$$

Lorentz symmetry of scattering in AFS is equivalent to global conformal symmetry $SL(2,\mathbb{C})$ of celestial amplitudes

$$\mathcal{L}_I \widetilde{\mathcal{A}}_n = \bar{\mathcal{L}}_I \widetilde{\mathcal{A}}_n = 0, \tag{173}$$

where $\mathcal{A}_n$ is an n-point celestial amplitude and

$$\mathcal{L}_I = \sum_{k=1}^{n} L_{I,k}, \quad \bar{\mathcal{L}}_I = \sum_{k=1}^{n} \bar{L}_{I,k}, \qquad I \in \{-1, 0, 1\} \tag{174}$$

Here $L_{I,k}$ corresponds to a Lorentz generator related to each particle k. It is nothing else but a global Ward identity for correlation functions of primaries in 2D CFT (107) that we derived in the first lecture. We are ready to start building holographic symmetry correspondence dictionary:

4D Asymptotically Flat Spacetime		**2D Celestial CFT**
Lorentz Invariance $SO(1,3)$	$\longleftrightarrow$	Global Conformal Symmetry $SL(2,\mathbb{C})$

To arrive to Poincaré symmetry, we also need to investigate how translation generators act on the celestial amplitudes. Bulk translation invariance implies that

$$\mathcal{P}_\mu \widetilde{\mathcal{A}}_n = 0, \quad \mathcal{P}_\mu \equiv \sum_{k=1}^{n} P_{\mu,k}. \tag{175}$$

Actually, each of the translational generators $P_{\mu,k}$ acts on their own massless particles as their weight-shifting operators. To see this is really straightforward, we can start with the momentum space action on a massless wavefunction $\phi_\Delta(x,\vec{z})$:

$$\begin{aligned} P_\mu \phi_\Delta(x,\vec{z}) &= -i\frac{\partial}{\partial x^\mu}\int_0^\infty d\omega\, \omega^{\Delta-1} e^{i\omega\hat{q}(\vec{z})\cdot x} = \int_0^\infty d\omega\, \omega^{\Delta-1}(-i)(i\omega\hat{q}_\mu) e^{i\omega\hat{q}(\vec{z})\cdot x} = \\ &= \hat{q}_\mu \int_0^\infty d\omega\, \omega^{(\Delta+1)-1} e^{i\omega\hat{q}(\vec{z})\cdot x} = \hat{q}_\mu \phi_{\Delta+1}(x,\vec{z}) \end{aligned} \tag{176}$$

We see that the action of translation generator act as a weight-shifting operator and carries the information about the direction of translation $\hat{q}^\mu$, even though the wavefunctions already do not depend on the energies ω, since they are integrated over.

We can encode the action of a weight-shifting operator as an exponential operator $e^{\partial_\Delta} f(\Delta) = f(\Delta+1)$, therefore action of translations generator on a celestial primary basis immediately reads as:

$$P_{\mu,k} = \epsilon_k \hat{q}_\mu(z_k,\bar{z}_k) e^{\partial_{\Delta_k}}, \tag{177}$$

where $\epsilon_k = \pm 1$ distinguishes between incoming and outgoing particles.

Therefore, when acting on a celestial amplitude, it shifts the scaling dimension of k-th particle:

$$P_k\widetilde{\mathcal{A}}(\Delta_1,\dots,\Delta_n) = \epsilon_k\hat{q}_k\widetilde{\mathcal{A}}(\Delta_1,\dots,\Delta_k+1,\dots,\Delta_n).$$

After summing over all the particles whose weights have been shifted, we conclude that the translation symmetry of the amplitude (175) implies the following constraint:

$$\widetilde{\mathcal{A}}_n(\Delta_1+1,\Delta_2,\dots,\Delta_n)+\widetilde{\mathcal{A}}_n(\Delta_1,\Delta_2+1,\dots,\Delta_n)+\cdots+\widetilde{\mathcal{A}}_n(\Delta_1,\dots,\Delta_n+1)=0. \tag{178}$$

This relation can be considered as momentum conservation law on celestial sphere for scalar massless scattering. Continuing our dictionary,

4D Asymptotically Flat Spacetime		**2D Celestial CFT**
Translation Invariance (massless)	$\longleftrightarrow$	Weight-Shifting Operators ($\Delta\to\Delta+1$)

For massive scalars, the cute little weight-shifting action is replaced by

$$P^\mu = \frac{m}{2}\left[\left(\partial_z\partial_{\bar z}q^\mu + \frac{\partial_z q^\mu\partial_z+\partial_{\bar z}q^\mu\partial_{\bar z}}{\Delta-1}+\frac{q^\mu\partial_z\partial_{\bar z}}{(\Delta-1)^2}\right)e^{-\partial_\Delta}+\frac{\Delta q^\mu}{\Delta-1}e^{\partial_\Delta}\right]. \tag{179}$$

and is determined by imposing the on-shell condition

$$P_\mu P^\mu = -m^2, \tag{180}$$

as well as the Poincaré algebra

$$[P_\mu,P_\nu]=0,\quad [M_{\mu\nu},P_\rho]=\eta_{\mu\rho}P_\nu-\eta_{\nu\rho}P_\mu. \tag{181}$$

Exercise 10. *Verify that P^μ for massive scalars indeed satisfies the on-shell condition and reproduces the Poincaré commutation relations.*

3.1.2 Constraints from Poincaré symmetry

(172) and (177) imply that any celestial 4-point function can be put into the form

$$\widetilde{\mathcal{A}}_4 = K_{h_i,\bar h_i}(z_i,\bar z_i)\delta(z-\bar z)f^{h_i,\bar h_i}(z,\bar z), \tag{182}$$

with kinematical factor

$$K_{h_i,\bar h_i}(z_i,\bar z_i)=\prod_{i<j=1}^{4} z_{ij}^{h/3-h_i-h_j}\bar z_{ij}^{\bar h/3-\bar h_i-\bar h_j},\qquad h=\sum_{i=1}^{4}h_i, \tag{183}$$

and 2D conformally invariant cross-ratios

$$z=\frac{z_{13}z_{24}}{z_{12}z_{34}},\qquad \bar z=\frac{\bar z_{13}\bar z_{24}}{\bar z_{12}\bar z_{34}}. \tag{184}$$

For massless scattering, (184) are related to the bulk variables (Mandelstam invariants) by

$$z=-\frac{t}{s},\qquad s=-(p_1+p_2)^2,\qquad t=-(p_1+p_3)^2. \tag{185}$$

Additionally, in this case momentum conservation (175) implies that

$$\sum_{j=1}^{4}K_{h_j+\frac12,\bar h_j+\frac12}(z_i,\bar z_i)f^{h_j+\frac12,\bar h_j+\frac12}(z,\bar z)=0. \tag{186}$$

Since the conformally covariant factor (183) is translationally invariant by itself, (186) can be non-trivially obeyed if

$$f^{h_i+\frac12,\bar h_i+\frac12}(z,\bar z)=f^{h_j+\frac12,\bar h_j+\frac12}(z,\bar z),\quad \forall i,j. \tag{187}$$

By induction it can be shown that (187) implies that

$$f^{h_i,\bar{h}_i}(z,\bar{z}) = f^{\beta,J_i}(z,\bar{z}), \quad \beta = \sum_{i=1}^{4}(h_i+\bar{h}_i) = \sum_{i=1}^{4}\Delta_i. \tag{188}$$

More generally, (175) will be obeyed by celestial amplitudes with both massive and massless particles, provided $P_{\mu,k}$ are chosen appropriately. For example, consider the three-point amplitude of two massless and one massive scalars computed in previous lecture. The same result can be perhaps more easily recovered by considering the constraint

$$\left(P_1+P_2+P_3^{(m)}\right)\widetilde{\mathcal{A}}_3(1,2,3^{(m)}) = 0, \tag{189}$$

where P_1, P_2 are the massless momentum generators in (177) while $P_3^{(m)}$ is the massive momentum (179). Global conformal invariance fixes

$$\widetilde{\mathcal{A}}(1,2,3^{(m)}) = \frac{C(\Delta_1,\Delta_2,\Delta_3)}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3}|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}|z_{13}|^{\Delta_1+\Delta_3-\Delta_2}}, \tag{190}$$

hence (189) leads to the following recursion relations on the 3-point coefficients [18]

$$\begin{aligned}
\left(\frac{\Delta_{12}^2}{4}-\frac{(\Delta_3-1)^2}{4}\right)C_{\Delta_1,\Delta_2,\Delta_3-1}+\Delta_3(\Delta_3-1)C_{\Delta_1,\Delta_2,\Delta_3+1}&=0,\\
4\epsilon_2(\Delta_3-1)C_{\Delta_1,\Delta_2+1,\Delta_3}+m\epsilon_3(\Delta_3-1-\Delta_{12})C_{\Delta_1,\Delta_2,\Delta_3-1}&=0,\\
4\epsilon_1(\Delta_3-1)C_{\Delta_1+1,\Delta_2,\Delta_3}+m\epsilon_3(\Delta_3-1+\Delta_{12})C_{\Delta_1,\Delta_2,\Delta_3-1}&=0,
\end{aligned} \tag{191}$$

where $\Delta_{12} \equiv \Delta_1 - \Delta_2$.

Exercise 11. *Show that* (191) *are solved by*

$$C(\Delta_1,\Delta_2,\Delta_3) = B\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2},\frac{\Delta_2-\Delta_1+\Delta_3}{2}\right)c_{\Delta_1,\Delta_2,\Delta_3}, \tag{192}$$

where

$$\begin{aligned}
c_{\Delta_1,\Delta_2,\Delta_3-1} &= c_{\Delta_1,\Delta_2,\Delta_3+1},\\
c_{\Delta_1+1,\Delta_2,\Delta_3} &= c_{\Delta_1,\Delta_2+1,\Delta_3},\\
c_{\Delta_1+1,\Delta_2,\Delta_3} &= c_{\Delta_1,\Delta_2,\Delta_3-1}.
\end{aligned} \tag{193}$$

The first constraint in (193) implies c is periodic in Δ_3 of period 2, the second implies periodicity of period 1 in Δ_1 and Δ_2 up to dependence of $\Delta_1+\Delta_2$ while additionally, the last constraint implies c is periodic of period 1 in Δ_1, Δ_3 up to dependence of $\sum_{i=1}^{3}\Delta_i$ in which case the period becomes 2. The periodic function is set to a constant by requiring the inverse Mellin transform to be well defined.

3.1.3 Action of supertranslations on $\mathcal{CS}$

An interesting question to explicitly write down is how supertranslations act on our space of celestial states. Remember firstly our generating vector field (10). To see it easier, we can write down the action of the leading in r term, changing coordinates fromm $(u,r,z,\bar{z})$ to $(t,r,z,\bar{z})$:

$$\xi = f(z,\bar{z})\partial_u = f(z,\bar{z})\partial_t. \tag{194}$$

Now we can see the action of it on wavefunctions in momentum space

$$\xi e^{ip\cdot x} = f(z,\bar{z})\partial_t e^{-i\omega t+i\vec{p}\cdot\vec{x}} = -i\omega f(z,\bar{z})e^{ip\cdot x}. \tag{195}$$

Again, we do a Mellin transform and obtain:

$$\xi\phi_\Delta(x,\vec{z}) = \int_0^\infty d\omega\,\omega^{\Delta-1}(-i\omega f(z,\bar{z}))e^{i\omega\hat{q}(\vec{z})\cdot x} = -if(z,\bar{z})\phi_{\Delta+1}(x,\vec{z}). \tag{196}$$

So supertranslations also act like weight-shifting operators. Again, if among them we can pick the $f(z,\bar{z})$ responsible for 4 global spacetime translations.

Exercise 12. *Retrieve the weight-shifting action* (176) *of global translations using* (12).

4D Asymptotically Flat Spacetime		**2D Celestial CFT**
Supertranslation Invariance (massless)	$\longleftrightarrow$	Weight-Shifting Operators ($\Delta \to \Delta+1$)

3.1.4 Action of superrotations on CS and Ward identities

What if instead of stopping, we proceeded in the large-r expansion of asymptotic metric (4)? Then, we would write down the next order term in the expansion:

$$
\begin{aligned}
ds^2 = & - du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 \\
& - 2U_z dudz - 2U_{\bar{z}}dud\bar{z} + \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right) du\, dz + \cdots , \qquad (197)
\end{aligned}
$$

$N_z(u, z, \bar{z})$ is known as the angular momentum aspect, because its integrals over the sphere, contracted with a rotational vector field, give the total angular momentum.

This angular momentum aspect is part of the constraint equation $G_{uz} = 8\pi G T^M_{uz}$. This equation gives the following constraint on N_z:

$$
\partial_u N_z = \frac{1}{4}\partial_z(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u\partial_u\partial_z m_B - T_{uz}, \qquad (198)
$$

where we defined

$$
T_{uz} \equiv 8\pi G \lim_{r\to\infty}\left[r^2 T^M_{uz}\right] - \frac{1}{4}\partial_z(C_{zz}N^{zz}) - \frac{1}{2}C_{zz}D_z N^{zz}. \qquad (199)
$$

We see that in contrast to m_B, which was connected to the energy density, the angular momentum aspect N_z is connected with momentum density T_{uz}. Also this relation shows us that as soon as m_B and C_{zz} are specified, then $\partial_u N_z$ is also fixed, which means the N_z itself is fixed up to a function depending on $(z, \bar{z})$. To fix this function we impose another matching condition:

$$
N_z(z, \bar{z})|_{\mathscr{I}^+_-} = N_z(z, \bar{z})|_{\mathscr{I}^-_+}. \qquad (200)
$$

This condition implies a second infinite set of conserved charges, which can be constructed from an arbitary vector field Y^z on a sphere and are defined as:

$$
Q^+_Y = \frac{1}{8\pi G}\int_{\mathscr{I}^+_-} d^2z(Y_{\bar{z}}N_z + Y_z N_{\bar{z}}) = \frac{1}{8\pi G}\int_{\mathscr{I}^-_+} d^2z(Y_{\bar{z}}N_z + Y_z N_{\bar{z}}) = Q^-_Y. \qquad (201)
$$

Lorentz Killing vectors are of the general form

$$
\zeta_Y = \left(1 + \frac{u}{2r}\right) Y^z\partial_z - \frac{u}{2r}D^{\bar{z}}D_z Y^z\partial_{\bar{z}} - \frac{1}{2}(u + r)D_z Y^z\partial_r + \frac{u}{2}D_z Y^z\partial_u + c.c., \qquad (202)
$$

where $(Y^z, Y^{\bar{z}})$ is a two-dimensional vector field on $\mathcal{CS}^2$. At null infinity ζ_Y simplifies to

$$
\zeta_Y|_{\mathscr{I}^+} = Y^z\partial_z + \frac{u}{2}D_z Y^z\partial_u + c.c.. \qquad (203)
$$

The claim is that for flat Minkowski space, if we take

$$
Y^z = 1, z, z^2, i, iz, iz^2, \qquad (204)
$$

then the six real vector fields ζ_Y generate the Lorentz transformations.

But what if we don't? What if we let Y^z to be a local vector field? Then Y^z is allowed to be any holomorphic function on a celestial sphere, and this condition is solved by $Y^z = z^n$, $n \in \mathbb{Z}$. This choice of conserving local fields instead of imposing global symmetry will have interesting consequences for us.

Let's repeat the analysis similar to the leading soft theorem, but this time interest ourselves in the subleading (ω^0) mode. Authours in [28] showed us that the subleading relation leads to

$$
\begin{aligned}
\langle \text{out}|Q^+_S\mathcal{S} - \mathcal{S}Q^-_S|\text{in}\rangle &= -\frac{i}{4\pi}\int d^2z D_z^3 Y^z\hat{\epsilon}_{\bar{z}\bar{z}}S^{(1)-}\langle\text{out}|\mathcal{S}|\text{in}\rangle \qquad (205) \\
&= -i\sum_{k\in\text{in,out}}\left(Y^{z_k}\partial_{z_k} - \frac{\omega_k}{2}D_{z_k}Y^{z_k}\partial_{\omega_k}\right)\langle\text{out}|\mathcal{S}|\text{in}\rangle. \qquad (206)
\end{aligned}
$$

We look closely at the term $Y^{z_k}\partial_{z_k}$ and remember that superrotation symmetry set $Y^{z_k} = z_k^n$, $n \in \mathbb{Z}$ for us. Now the time has come to recall everything we learned about 2D CFT. The first term in the integral is nothing else but the Witt generators (90).

Rigorosity Alert 4. *Even though it is formally the Witt algebra, in most of the celestial literature one will see it being called the Virasoro algebra. For a reference, the Virasoro algebra is called a central extension of the Witt algebra and its generators obey the modified commutation relation*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

When the central charge $c=0$ the two algebras indeed are the same thing. At the tree level we will carelessly call it either the Witt or the Virasoro algebra, meaning the Virasoro without central extension. In the last lecture we will mention the $w_{1+\infty}$ algebra that will make everything so much more complicated that this notice will disappear in the ocean of pain.

If now we set

$$Y^{z_k} = \frac{1}{z-z_k} \tag{207}$$

and define stress-energy tensor component

$$T_{zz} \equiv i\int d^2w \frac{1}{z-w} D_w^2 D^{\bar{w}} N^{(1)}_{\bar{w}\bar{w}}, \tag{208}$$

then our Ward identity (206) will take form:

$$\langle T_{zz}\mathcal{O}_1\dots\mathcal{O}_n\rangle = \sum_{k=1}^{n}\left[\frac{\hat{h}_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}\hat{h}_k}{z-z_k} + \frac{1}{z-z_k}\partial_{z_k}\right]\langle\mathcal{O}_1\dots\mathcal{O}_n\rangle. \tag{209}$$

This should seem familiar.

Up to a confusing entrance of Christoffel symbol Γ^z_{zz} (which arises because of appearance of a curved background), it is nothing else but a standard 2D CFT Ward identity (106) in the stress-energy formulation from the first lecture. To be more precise, this is a Ward identity in 2D CFT on a curved background [29]. Therefore,

4D Asymptotically Flat Spacetime		**2D Celestial CFT**
Superrotation Invariance	$\longleftrightarrow$	Local Conformal Symmetry (Virasoro)

3.1.5 Closing remarks on the symmetry correspondence

This is how asymptotic symmemtries are connected to conformal symmetries on $\mathcal{CS}$. We started from the global Poincare symmetry of the spacetime and then built a generalized group of asymptotic symmetries called extended BMS group. This symmetry group consists of supertranslations and superrotations, and they have global translations and Lorentz transformations as a subset. To complete the story about symmetries, we provide the full table of their correspondence below:

4D Asymptotically Flat Spacetime		**2D Celestial CFT**
Lorentz Invariance $SO(1,3)$	$\longleftrightarrow$	Global Conformal Symmetry $SL(2,\mathbb{C})$
Superrotation Invariance	$\longleftrightarrow$	Local Conformal Symmetry (Virasoro)
Translation & Supertranslation Invariance	$\longleftrightarrow$	Weight-Shifting Operators ($\Delta \to \Delta+1$)

3.2 Shadow transform

In this section we will show another interesting property of conformal primary wavefunctions introducing shadow transform. Namely, we will show that the conformal primary wavefunction $\phi^{\pm}_{\Delta}$ is the shadow transform of $\phi^{\pm}_{2-\Delta}$.

Definition 9. *Given a 2-dimensional scalar conformal primary operator $\mathcal{O}_\Delta(\vec{w})$, its* ***shadow*** *[30–32] $\widetilde{\mathcal{O}}_\Delta(\vec{w})$ is a non-local operator defined as*

$$\widetilde{\mathcal{O}}_\Delta(\vec{w}) \equiv \frac{(\Delta-1)}{\pi}\int d^2\vec{w}' \frac{1}{|\vec{w}-\vec{w}'|^{2(2-\Delta)}}\mathcal{O}_\Delta(\vec{w}'). \tag{210}$$

The shadow operator $\widetilde{\mathcal{O}}_\Delta$ transforms as a scalar conformal primary operator with conformal dimension $2-\Delta$. The normalization constant is chosen so that our conformal primary wavefunctions transform nicely under (210).

To study the shadow transform of our conformal primary wavefunction, let us first note a useful identity (see, for example, [33])

$$\int d^2\vec{z}\frac{1}{|\vec{z}-\vec{w}|^{2(2-\Delta)}}\frac{1}{(-q(\vec{z})\cdot X)^\Delta}=\frac{\pi\Gamma(\Delta-1)}{\Gamma(\Delta)}\frac{(-X^2)^{1-\Delta}}{(-q(\vec{w})\cdot X)^{2-\Delta}}\,. \tag{211}$$

Notice that shadow transform of our conformal primary wavefunction directly follows from this identity:

$$\widetilde{\phi^\pm_\Delta}(X;\vec{w})=\phi^\pm_{2-\Delta}(X;\vec{w}). \tag{212}$$

Exercise 13. *Show this in detail.*

Remember now when we were taking the massless limit of bulk-to-boundary propagator (152), we obtained the expression for massless conformal primary wavefunction (153). I will rewrite the expansion of propagator below so we will have it close:

$$G_\Delta(y,\vec{w};\vec{z})=\frac{\pi}{\Delta-1}y^{2-\Delta}\delta^{(2)}(\vec{z}-\vec{w})+\frac{y^\Delta}{|\vec{z}-\vec{w}|^{2\Delta}}+\cdots \tag{213}$$

During this transition I deceived you a bit, cause if we substitute in the expression for the wavefunction both of these terms, the first one will, indeed, give a clear Mellin transform $\phi_\Delta(x,\vec{z})$, but the second term will be nothing else but shadow transform of it:

$$\psi^\pm_\Delta(x;\vec{z})\xrightarrow{m\to 0}\left(\frac{m}{2}\right)^{-\Delta}\frac{\pi}{\Delta-1}\int_0^\infty d\omega\,\omega^{\Delta-1}e^{\pm i\omega q(\vec{z})\cdot x} \tag{214}$$

$$+\left(\frac{m}{2}\right)^{\Delta-2}\int d^2\vec{w}\frac{1}{|\vec{w}-\vec{z}|^{2\Delta}}\int_0^\infty d\omega\,\omega^{1-\Delta}e^{\pm i\omega q(\vec{w})\cdot x}. \tag{215}$$

Let us rewrite the second term more carefully, substituting new scale dimension $\Delta'=2-\Delta$, so it will be more clear:

$$\int d^2\vec{w}\frac{1}{|\vec{w}-\vec{z}|^{2(2-\Delta')}}\underbrace{\int_0^\infty d\omega\,\omega^{\Delta'-1}e^{\pm i\omega q(\vec{w})\cdot x}}_{\phi_{\Delta'}(x,\vec{z})}=\widetilde{\phi_{\Delta'}}(x,\vec{z}), \tag{216}$$

Thus the conformal primary wavefunctions $\phi^\pm_\Delta$ and $\phi^\pm_{2-\Delta}$ should not be counted as linearly independent solutions to the massive Klein-Gordon equation as they are related by a shadow transform, which means that we can disregard the second term, as it is not a linearly independent wavefunction.

4 Lecture 4. Celestial OPEs, $w_{1+\infty}$ algebra of soft gravitons and its connection to twistors

4.1 OPEs from asymptotic symmetries

In this section we will consider some celestial OPEs that originate from analysis of asymptotic symmetries. In the next section we will see that such analysis gives us the same result as scattering amplitude approach of collinear limits. The following will be based on an article [34] that discusses gluon and graviton OPEs. However, we will interest ourselves in the graviton case (since our school is called School on *Gravitational* Physics).

When two massless particles p_1, p_2 interact through a 3-vertex, the resulting propagator will be proportional to $\frac{1}{p_1 \cdot p_2}$. Remembering also the celestial parametrization $p_1 \cdot p_2 = -\omega_1\omega_2 z_{12}\bar{z}_{12}$, we see that such a pole corresponds to at most a simple pole at $z_1 = z_2$. We will consider holomorphic singularities in z_{12}, since antiholomorphic case is analogous. Schematically the OPE of conformal primaries takes the form

$$\mathcal{O}_{\Delta_1,s_1}(z_1,\bar{z}_1)\mathcal{O}_{\Delta_2,s_2}(z_2,\bar{z}_2) \propto \frac{1}{z_{12}}\mathcal{O}_{\Delta_3,s_3}(z_2,\bar{z}_2) + \text{order}(z_{12}^0). \tag{217}$$

According to [34], the following constraint for these scaling dimensions arising from vertex dimensions should be satisfied for graviton OPEs:

$$\Delta_3 = \Delta_1 + \Delta_2. \tag{218}$$

We will denote by $G^{\pm}$ an operator corresponding to a graviton of helicity $s = \pm 2$ that crosses the celestial sphere at a point $(z, \bar{z})$. Notice that OPE G^-G^- is nonsingular when $z_{12} \to 0$, it has only singularities in $\bar{z}_{12}$. This leads to an OPE of the form

$$G^+_{\Delta_1}(z_1,\bar{z}_1)G^{\pm}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{\bar{z}_{12}}{z_{12}}E_{\pm}(\Delta_1,\Delta_2)G^{\pm}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2) \tag{219}$$

for some coefficitents $E_+(\Delta_1,\Delta_2) = E_+(\Delta_2,\Delta_1)$ and $E_-(\Delta_1,\Delta_2)$. After imposing leading and subsubleading soft theorems, we can find the actual form of these coefficients, which are Euler beta functions:

$$E_{\pm}(\Delta_1,\Delta_2) = -\frac{\kappa}{2}B(\Delta_1 - 1, \Delta_2 \mp 2 + 1). \tag{220}$$

4.2 Celestial OPE's and collinear limit

While the celestial OPE's are somewhat cumbersome, one still can study some of their properties without specifying their precise form. Here we will pay attention to a so-called *collinear limit* of two particles participating in a scattering process.

$$\frac{1}{p_1 \cdot p_2} = \frac{1}{E_1E_2 - \vec{p}_1 \cdot \vec{p}_2} = \frac{1}{E_1E_2 - |p_1||p_2|\cos\theta} = \frac{1}{E_1E_2(1-\cos\theta)}. \tag{221}$$

Therefore, when $p_1||p_2$ and $\cos\theta = 1$ the propagator will have a divergence in momentum space.

Firstly collinear limit was obtained for gluons in [35]. As mentioned, we will focus on gravitons in this lecture. So, if we consider a tree-level n-graviton scattering amplitude, the limit when $z_{12} \to 0$ will behave as:

$$\lim_{z_{ij}\to 0}\mathcal{A}^n_{s_1\ldots s_n}(p_1\ldots p_n) = \sum_{s=\pm 2}\text{Split}^s_{s_is_j}(p_i,p_j)\mathcal{A}^{n-1}_{s_1\ldots s\ldots s_n}(p_1\ldots P\ldots p_n), \tag{222}$$

where $P^\mu = p_i^\mu + p_j^\mu$, $\omega_P = \omega_i + \omega_j$ in collinear limit and the collinear factor $\text{Split}^s_{s_is_j}(p_i,p_j)$ takes the form [36]:

$$\text{Split}^2_{22}(p_i,p_j) = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\frac{\omega_P^2}{\omega_i\omega_j}, \qquad \text{Split}^{-2}_{2-2}(p_i,p_j) = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\frac{\omega_j^3}{\omega_i\omega_P^2}, \tag{223}$$

with all other combinations of helicities vanishing. In the collinear limit, the celestial gravity amplitude $\widetilde{\mathcal{A}}$ becomes

$$\begin{aligned}&\widetilde{\mathcal{A}}_{s_1\cdots s_n}(\Delta_1,z_1,\bar{z}_1,\cdots,\Delta_n,z_n,\bar{z}_n) \xrightarrow{i\|j} \\ &\prod_{k=1}^n\int_0^\infty d\omega_k\,\omega_k^{\Delta_k-1}\sum_{s=\pm 2}\text{Split}^s_{s_is_j}(p_i,p_j)\mathcal{A}_{s_1\cdots s\cdots s_n}(p_1,\cdots,P,\cdots,p_n) + \cdots.\end{aligned} \tag{224}$$

To simplify, we make the following change of variables

$$\omega_i = t\omega_P, \qquad \omega_j = (1-t)\omega_P, \tag{225}$$

so that for example

$$\int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1} \int_0^\infty d\omega_j\, \omega_j^{\Delta_j-1} \mathrm{Split}^2_{22}(p_i,p_j) = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}} \int_0^1 dt\, t^{\Delta_i-2}(1-t)^{\Delta_j-2} \int_0^\infty d\omega_P\, \omega_P^{\Delta_i+\Delta_j-1}. \tag{226}$$

The t integral is immediately recognizable as the integral representation of the Euler beta function,

$$B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1}, \tag{227}$$

whose origin is hence a splitting factor for the conformal weight between the two collinear external particles. Since the only t dependence on the right hand side of (224) comes from $\mathrm{Split}^s_{s_is_j}(p_i,p_j)$, one finds

$$\begin{aligned} &\lim_{z_{ij}\to 0} \widetilde{\mathcal{A}}_{s_1\cdots 2\cdots 2\cdots}(\Delta_1,z_1,\bar{z}_1,\cdots,\Delta_i,z_i,\bar{z}_i,\cdots,\Delta_j,z_j,\bar{z}_j,\cdots) \to \\ &\quad -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}} B(\Delta_i-1,\Delta_j-1)\widetilde{\mathcal{A}}_{s_1\cdots 2\cdots}(\Delta_1,z_1,\bar{z}_1,\cdots,\Delta_i+\Delta_j,z_j,\bar{z}_j,\cdots) + \mathrm{order}(z_{ij}^0). \end{aligned} \tag{228}$$

Since this holds in any celestial amplitude, it implies the leading OPE between two positive helicity gravitons is

$$G^+_{\Delta_1}(z_1,\bar{z}_1)G^+_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}} B(\Delta_1-1,\Delta_2-1) G^+_{\Delta_1+\Delta_2}(z_2,\bar{z}_2), \tag{229}$$

in agreement with (219). By similar arguments, one also finds the following leading OPE between opposite helicity gravitons

$$G^+_{\Delta_1}(z_1,\bar{z}_1)G^-_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}} B(\Delta_1-1,\Delta_2+3) G^-_{\Delta_1+\Delta_2}(z_2,\bar{z}_2), \tag{230}$$

again in agreement with (219).

4.3 $w_{1+\infty}$ algebra

After getting familiar with graviton OPEs, we can try to see how these operators $G_\Delta(z,\bar{z})$ generate a tower of soft theorems, order by order. This section is based on article [37]. So, firstly we define what is a primary operator corresponding to a soft positive-helicity graviton:

$$H^k = \lim_{\varepsilon\to 0} \varepsilon G^+_{k+\varepsilon}, \quad k = 2,1,0,-1,\ldots, \tag{231}$$

where weights are given by

$$(h,\bar{h}) = \left(\frac{k+2}{2}, \frac{k-2}{2}\right). \tag{232}$$

To transform OPE to a commutator, for holomorphic objects we define:

$$[A,B](z) = \oint_z \frac{dw}{2\pi i} A(w)B(z), \tag{233}$$

The soft current algebra found in [38] for gravity is

$$[H^k_m, H^l_n] = -\frac{\kappa}{2}[n(2-k)-m(2-l)] \frac{(\frac{2-k}{2}-m+\frac{2-l}{2}-n-1)!}{(\frac{2-k}{2}-m)!(\frac{2-l}{2}-n)!} \frac{(\frac{2-k}{2}+m+\frac{2-l}{2}+n-1)!}{(\frac{2-k}{2}+m)!(\frac{2-l}{2}+n)!} H^{k+l}_{m+n}, \tag{234}$$

To write the algebra (234) in a simpler form we define

$$w^p_n = \frac{1}{\kappa}(p-n-1)!(p+n-1)! H^{-2p+4}_n. \tag{235}$$

This is essentially the light-transform or right-shadow [39] in one dimension and p is the right-shadowed weight. Relation (234) then simply becomes

$$[w^p_m, w^q_n] = [m(q-1)-n(p-1)] w^{p+q-2}_{m+n}. \tag{236}$$

The index k on H^k_m runs over $k = 2, 1, 0, \ldots$, while p runs over the positive half integral values

$$p = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \tag{237}$$

The restriction $\bar{h} \leq m \leq -\bar{h}$ or

$$\frac{k-2}{2} \leq m \leq \frac{2-k}{2}$$

becomes

$$1 - p \leq m \leq p - 1. \tag{238}$$

The commutators (236) were first written down by Ioannis Bakas in 1989 [40] and corresponds to an infinite dimensional algebra of the area-preserving diffeomorphisms of the 2-plane. In this work m is an arbitrary integer and the resulting algebra is now referred to as $w_{1+\infty}$. The closed $p = 2$ subalgebra of $w_{1+\infty}$ is the $c = 0$ Virasoro algebra. This algebra in (236) is a subgroup of $w_{1+\infty}$ with m restricted according to (238). The resulting restricted algebra is known as the *wedge subalgebra of* $w_{1+\infty}$, and can also be rewritten as $GL(\infty, \mathbb{R})$ [41].

Moreover, each $w^p_m(z)$ acts as a current on the celestial sphere. Hence we have a $GL(\infty, \mathbb{R})$ Kac-Moody algebra. The modes of the Kac-Moody currents act on the states of the 2D CCFT associated by the state-operator correspondence to operator insertions in the celestial sphere [42]. A review of this and other fascinating aspects of W algebras can be found in [43].

The closed subgroup generated by

$$L_m \equiv w^2_m, \quad m = -1, 0, 1 \tag{239}$$

is an '$SL(2, \mathbb{R})_w$' current algebra implied by the subleading soft graviton theorem. It is related to the original generators in (234) by

$$L_0 = \frac{1}{\kappa} H^0_0, \quad L_{\pm 1} = \frac{2}{\kappa} H^0_{\pm 1}. \tag{240}$$

w^q_n transforms under this $SL(2, \mathbb{R})_w$ as

$$[L_m, w^q_n] = [m(q-1) - n] w^q_{m+n}. \tag{241}$$

like the modes of an $SL(2, \mathbb{R})_w$ primary operator of $SL(2, \mathbb{R})_w$ weight q. Since $q \geq 1$ here, these are all positive. However, unlike the H^k_n, they do not transform canonically under the original $SL(2, \mathbb{R})_R$ because of the mode-dependent relation between the two. Indeed the normalization of the H^k_n modes was chosen so that they transform canonically under the original $SL(2, \mathbb{R})_R$ conformal generators with weights $\bar{h} = \frac{k-2}{2}$.

Evidently $SL(2, \mathbb{R})_R$ and $SL(2, \mathbb{R})_\mathrm{w}$ are not quite the same thing. w^p_n on the LHS of (235) lies in a positive weight $h_w = p$ representation of $SL(2, \mathbb{R})_\mathrm{w}$, while H^{-2p+4}_n on the RHS lies in a $2p + 1$ dimensional, negative weight $\bar{h} = 1 - p$ representation of $SL(2, \mathbb{R})_R$. (235) is the relation between them. Both representations are $2p + 1$ dimensional, because finite weight h $SL(2, \mathbb{R})$ representations are $2h - 1$ dimensional for positive half integral h and $-2h - 1$ dimensional for negative half integral h.

Significantly, the $SL(2, \mathbb{R})_\mathrm{w}$ representations which appear here are all positive weight.

4.4 Connections with twistors and string theory

The statement of paper [44] is that $w_{1+\infty}$ celestial symmetry arises somewhat naturally on twistor space as a Poisson structure. I will try to give a brief review of what I've understood from this paper.

We start by introducing twistor space. I will not be very matematically rigorous right now, for further read about twistors I refer to lecture notes [45].

Step 1: We start from a 4-vector $x^\mu = (x^0, x^1, x^2, x^3) \in \mathbb{R}^{1,3}$ in Minkowski spacetime with metric

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{242}$$

Step 2: We complexify the vector $\mathbb{R}^{1,3} \to \mathbb{C}^4$ by simply making each of its components a complex number instead of a real one, while preserving the metric in the aforementioned form.

Step 3: Out of this complex vector we compose a complex matrix:

$$x^{\alpha\dot{\alpha}} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} x_{\beta\dot{\beta}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 - x^3 & -(x^1 + \mathrm{i}x^2) \\ -x^1 + \mathrm{i}x^2 & x^0 + x^3 \end{pmatrix}. \tag{243}$$

which indices $\alpha, \beta, \dot{\alpha}, \dot{\beta} \in \{0, 1\}$ we raise and lower by Levi-Civita symbols $\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\dot{\alpha}\dot{\beta}}$ according to $a_\alpha \equiv a^\beta \epsilon_{\beta\alpha}$ and $b^\alpha \equiv \epsilon^{\alpha\beta} b_\beta$.

Step 4: We find a *twistor space* described by 4 coordinates Z^A to be connected with spinor variables as follows:

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha), \quad \mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\lambda_\alpha. \tag{244}$$

This is enough twistors for today.

When the spacetime is not flat anymore, the twistor construction is not that straightforward, because we need to deform the twistor space itself and introduce some structures corresponding to gravitons [46]. We will not discuss this construction in details.

Now we will give mathematical motivation of why to search for $w_{1+\infty}$ in twistor space. Let $\mu^{\dot{\alpha}} = (\mu^{\dot{0}}, \mu^{\dot{1}})$ be coordinates on the complex plane, with Poisson structure

$$\{f, g\} := \epsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial f}{\partial \mu^{\dot{\alpha}}} \frac{\partial g}{\partial \mu^{\dot{\beta}}} = \frac{\partial f}{\partial \mu^{\dot{0}}} \frac{\partial g}{\partial \mu^{\dot{1}}} - \frac{\partial f}{\partial \mu^{\dot{1}}} \frac{\partial g}{\partial \mu^{\dot{0}}} -, \tag{245}$$

If we now construct polynomials:

$$w^p_m := (\mu^{\dot{0}})^{p+m-1}(\mu^{\dot{1}})^{p-m-1}, \qquad |m| \leq p-1, \tag{246}$$

so that $p \pm m - 1 \in \mathbb{Z}_{\geq 0}$, then the Poisson bracket acting on these elements gives

$$\{w^p_m, w^q_n\} = 2(m(q-1) - n(p-1))w^{p+q-2}_{m+n}. \tag{247}$$

This defines the commutation relations of the basis elements w^p_m of $w_{1+\infty}$, the same as we obtained for soft gravitons in (236).

This hints us that the algebra of soft gravitons appears naturally when we go to complex coordinates and sequently to twistor space. However, to see it clearly, which article [44] does, we need to introduce an action (twistor sigma model action) and the corresponding hamiltonians g representing soft gravitons. We will not perform it all in the current lecture, but will provide the result which follows from the authors' analysis.

These hamiltonians can be decomposed into modes and will obey the same Poisson brackets relation as purely abstract structure (247), reproducing the result obtained from graviton OPEs:

$$\{g^p_{m,r}, g^q_{n,s}\} = 2(m(q-1) - n(p-1))g^{p+q-2}_{m+n,r+s}. \tag{248}$$

4.5 Remember...

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